MATHEMATICAL FORMULAS FOR PROBLEMS IN "BOLIB 2019: BILEVEL OPTIMIZATION LIBRARY OF TEST PROBLEMS VERSION 2"

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ABSTRACT. This document is a supplementary material for the Bilevel Optimization LIBrary of test problems (BOLIB–for short), which contains a collection of test problems to help support the development of numerical solvers for bilevel optimization. It contains the mathematical formulas of all the problems in BOLIB together with some useful information on the corresponding solutions.

CONTENTS

The examples presented in this document are for a bilevel optimization problem of the form

$$\begin{array}{ll} \min & F(x,y) \\ \text{s.t.} & G(x,y) \le 0, \\ & y \in S(x) := \arg\min\limits_{y} \{f(x,y): \ g(x,y) \le 0\}, \end{array} \tag{P}$$

where the functions $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_G}$ and $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_g}$ describe the upper-level and lower-level constraints, respectively. On the other hand, $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ (resp. $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$) denotes the upper-level (resp. lower-level) objective function. The set-valued map $S : \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_y}$ represents the optimal solution set-valued mapping of the lower-level problem. Below, we provide formulas of the functions F, G, f, and g involved in problem (P) for all examples included in the BOLIB Version 2 library, together with true or best known values of the solutions, and some useful background information in some cases.

1. NONLINEAR BILEVEL EXAMPLES

Problem name: `AiyoshiShimizu1984Ex2`

Source: [1]

Date: December 13, 2019.
The work of the first and second authors is funded by the EPSRC Grant EP/P022553/1.

Description: AiyoshiShimizu1984Ex2 is defined as follows

$$\begin{aligned}
F(x,y) &:= 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60\\
G(x,y) &:= \begin{bmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ x - 50_2 \\ -x \end{bmatrix}\\
f(x,y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2\\
g(x,y) &:= \begin{bmatrix} 2y - x + 10_2 \\ -y - 10_2 \\ y - 20_2 \end{bmatrix}
\end{aligned}$$

Comment: Here, we write $50_2 := (50, 50)^\top$. Similar rule is applied into $10_2, 20_2$ and the context in the whole manuscript. The global optimal solution of the problem is $(25, 30, 5, 10)$ according to [1]. A local optimal one is $(0, 0, -10, -10)$ by [28].

Problem name: `AllendeStill2013`

Source: [2]

Description: AllendeStill2013 is defined as follows

$$\begin{aligned}
F(x,y) &:= x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2\\
G(x,y) &:= \begin{bmatrix} -x \\ -y \\ x_1 - 2 \end{bmatrix}\\
f(x,y) &:= y_1^2 - 2x_1y_1 + y_2^2 - 2x_2y_2\\
g(x,y) &:= \begin{bmatrix} (y_1 - 1)^2 - 0.25 \\ (y_2 - 1)^2 - 0.25 \end{bmatrix}
\end{aligned}$$

Comment: The global optimal solution from [2] is $(0.5, 0.5, 0.5, 0.5)$.

Problem name: `AnEtal2009`

Source: [3]

Description: AnEtal2009 is defined as follows

$$\begin{aligned}
F(x,y) &:= \tfrac{1}{2}(x^\top, y^\top)H(x^\top, y^\top)^\top + c_1^\top x + c_2^\top y\\
G(x,y) &:= \begin{bmatrix} -x \\ -y \\ Ax + By + d \end{bmatrix}\\
f(x,y) &:= y^\top Px + \tfrac{1}{2}y^\top Qy + q^\top y\\
g(x,y) &:= Dx + Ey + b
\end{aligned}$$

with H, c_1, c_2, A, B, d, P, Q, q, D, E, and b, respectively defined as follows

$$H := \begin{bmatrix} -3.8 & 4.4 & 1.2 & -2.2 \\ 4.4 & -2.2 & 0.6 & 1.8 \\ 1.2 & 0.6 & 0.0 & 0.4 \\ -2.2 & 1.8 & 0.4 & 0.0 \end{bmatrix}, \quad c_1 := \begin{bmatrix} 935.74474 \\ 87.53654 \end{bmatrix}, \quad c_2 := \begin{bmatrix} 121.96196 \\ 299.24825 \end{bmatrix}$$

$$A := \begin{bmatrix} 0.00000 & 3.88889 \\ -2.00000 & 8.77778 \end{bmatrix}, \quad B := \begin{bmatrix} 4.88889 & 7.44444 \\ -5.11111 & 0.88889 \end{bmatrix}, \quad d := \begin{bmatrix} -61.57778 \\ -0.80000 \end{bmatrix}$$

$$P := \begin{bmatrix} -17.85000 & 6.57500 \\ 30.32500 & 30.32500 \end{bmatrix}, \quad Q := \begin{bmatrix} 21.10204 & 11.81633 \\ -5.11111 & -14.44898 \end{bmatrix}, \quad q := \begin{bmatrix} -18.21053 \\ 13.05263 \end{bmatrix}$$

$$D := \begin{bmatrix} 5.00000 & 7.44444 \\ -8.33333 & 3.00000 \\ -8.66667 & -8.55556 \\ 6.44444 & -5.11111 \end{bmatrix}, \quad E := \begin{bmatrix} 3.88889 & 1.77778 \\ 6.88889 & 6.11111 \\ -5.33333 & -7.00000 \\ 1.44444 & 4.44444 \end{bmatrix}, \quad b := \begin{bmatrix} -39.62222 \\ -60.00000 \\ 72.37778 \\ -17.28889 \end{bmatrix}$$

Comment: $(0.200001, 1.999997, 3.999998, 4.600005)$ is the global optimal solution of the problem; cf. [3].

Problem name: `Bard1988Ex1`

Source: [4]

Description: Bard1988Ex1 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^2 + (2y+1)^2 \\ G(x,y) &:= -x \\ f(x,y) &:= (y-1)^2 - 1.5xy \\ g(x,y) &:= \begin{bmatrix} -3x + y + 3 \\ x - 0.5y - 4 \\ x + y - 7 \\ -y \end{bmatrix} \end{aligned}$$

Comment: $(1,0)$ is the global optimum and $(5,2)$ is a local optimal point.

Problem name: `Bard1988Ex2`

Source: [4]

Description: Bard1988Ex2 is defined as follows

$$\begin{aligned} F(x,y) &:= (200 - y_1 - y_3)(y_1 + y_3) + (160 - y_2 - y_4)(y_2 + y_4) \\ G(x,y) &:= \begin{bmatrix} x_1 + x_2 + x_3 + x_4 - 40 \\ -[10, 5, 15, 20]^\top + x \\ -x \end{bmatrix} \\ f(x,y) &:= (y_1 - 4)^2 + (y_2 - 13)^2 + (y_3 - 35)^2 + (y_4 - 2)^2 \\ g(x,y) &:= \begin{bmatrix} 0.4y_1 + 0.7y_2 - x_1 \\ 0.6y_1 + 0.3y_2 - x_2 \\ 0.4y_3 + 0.7y_4 - x_3 \\ 0.6y_3 + 0.3y_4 - x_4 \\ -[20, 20, 40, 40]^\top + y \\ -y \end{bmatrix} \end{aligned}$$

Comment: This version of the problem is taken from [11]. The original one in [4] has two lower-level problem. The upper-and lower-level optimal value are respectively obtained as -6600.00 and 57.48 in the former paper. However, the global upper-and lower-level optimal value should be -6600.00 and 54 according to [4].

Problem name: `Bard1988Ex3`

Source: [4]

Description: `Bard1988Ex3` is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & -x_1^2 - 3x_2 - 4y_1 + y_2^2 \\
G(x,y) & := & \begin{bmatrix} x_1^2 + 2x_2 - 4 \\ -x \end{bmatrix} \\
f(x,y) & := & 2x_1^2 + y_1^2 - 5y_2 \\
g(x,y) & := & \begin{bmatrix} -x_1^2 + 2x_1 - x_2^2 + 2y_1 - y_2 - 3 \\ -x_2 - 3y_1 + 4y_2 + 4 \\ -y \end{bmatrix}
\end{array}
$$

Comment: The global upper-and lower-level optimal objective value are respectively obtained as -12.68 and -1.02 in the paper [8].

Problem name: `Bard1991Ex1`

Source: [5]

Description: Bard1991Ex1 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & x + y_2 \\
G(x,y) & := & \begin{bmatrix} -x + 2 \\ x - 4 \end{bmatrix} \\
f(x,y) & := & 2y_1 + xy_2 \\
g(x,y) & := & \begin{bmatrix} x - y_1 - y_2 + 4 \\ -y \end{bmatrix}
\end{array}
$$

Comment: The global optimal solution of the problem is $(2, 6, 0)$; cf. [5].

Problem name: `BardBook1998`

Source: [6]

Description: BardBook1998 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\
G(x,y) & := & \begin{bmatrix} x - 50_2 \\ -x \end{bmatrix} \\
f(x,y) & := & 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60 \\
g(x,y) & := & \begin{bmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ 2y - x + 10_2 \\ y - 20_2 \\ -y - 10_2 \end{bmatrix}
\end{array}
$$

Comment: The global optimal solution is $(25, 30, 5, 10)$.

Problem name: `CalamaiVicente1994a`

Source: [7]

Description: CalamaiVicente1994a is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & \frac{1}{2}(x-1)^2 + \frac{1}{2}y^2 \\
f(x,y) & := & \frac{1}{2}y - xy \\
g(x,y) & := & \begin{bmatrix} x-y-1 \\ -x-y+1 \\ x+y-\rho \end{bmatrix}
\end{array}
$$

Comment: It is assumed in [7] that the parameter $\rho \geq 1$. We consider the following scenarios studied in the latter reference:

(i) For $\rho = 1$, the point $(1, 0)$ is global optimum of the problem.
(ii) For $1 < \rho < 2$, the point $\frac{1}{2}(1+\rho, -1+\rho)$ is a global optimal solution, while $\frac{1}{2}(1, 1)$ is a local optimal solution of the problem.
(iii) For $\rho = 2$, the points $\frac{1}{2}(1, 1)$ and $\frac{1}{2}(3, 1)$ are global optimal solution.
(iv) For $\rho > 2$, the point $\frac{1}{2}(1, 1)$ is global optimum of the problem.

Problem name: `CalamaiVicente1994b`
Source: [7]
Description: CalamaiVicente1994b is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & \frac{1}{2}\sum_{i=1}^{4}(x_i-1)^2 + \sum_{i=1}^{2} y_i^2 \\
f(x,y) & := & \sum_{i=1}^{2}\left(\frac{1}{2}y_i^2 - x_i y_i\right) \\
g(x,y) & := & \begin{bmatrix} x-y-1_2 \\ -x-y+1_2 \\ x_1+y_1-1.5 \\ x_1+y_2-3 \end{bmatrix}
\end{array}
$$

Comment: The global optimal solution is $(1.25, 0.5, 1, 1, 0.25, 0.5)$ with upper-and lower-level optimal objective value being 0.3125 and -0.4063.

Problem name: `CalamaiVicente1994c`
Source: [7]
Description: CalamaiVicente1994c is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & \frac{1}{2}x^\top Ax + \frac{1}{2}y^\top By + a^\top x + 2 \\
f(x,y) & := & \frac{1}{2}y^\top By + x^\top Cy \\
g(x,y) & := & Dx + Ey + d
\end{array}
$$

with A, B, a, B and C respectively as follows

$$A := \begin{bmatrix} 197.2 & 32.4 & -129.6 & -43.2 \\ 32.4 & 110.8 & -43.2 & -14.4 \\ -129.6 & -43.2 & 302.8 & -32.4 \\ -43.2 & -14.4 & -32.4 & 289.2 \end{bmatrix}, \quad B := \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, \quad a := \begin{bmatrix} -8.56 \\ -9.52 \\ -9.92 \\ -16.64 \end{bmatrix}$$

$$C := \begin{bmatrix} -132.4 & -10.8 \\ -10.8 & -103.6 \\ 43.2 & 14.4 \\ 14.4 & 4.8 \end{bmatrix}, \quad D := \begin{bmatrix} 13.24 & 1.08 & -4.32 & -1.44 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ 13.24 & 1.08 & -4.32 & -1.44x_4 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ -13.24 & -1.08 & +4.32 & +1.44 \\ -1.08 & -10.36 & +1.44 & +0.48 \end{bmatrix}$$

$$E := \begin{bmatrix} -10 & 0 \\ 0 & -10 \\ 10 & 0 \\ 0 & 10 \\ -10 & 0 \\ 0 & -10 \end{bmatrix}, \quad d := \begin{bmatrix} -1 \\ -1 \\ -1.5 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

Comment: According to [7], the problem has a unique global optimal solution

$$(0.13085,\ 0.05195,\ 0.1022,\ 0.0674,\ 0.025,\ 0.05)$$

with the corresponding upper-level and lower-level objective function values being 0.3125 and -0.4063..

Problem name: `CalveteGale1999P1`

Source: [9]

Description: CalveteGale1999P1 is defined as follows

$$\begin{aligned} F(x,y) &:= -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3 \\ G(x,y) &:= -x \\ f(x,y) &:= \frac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3} \\ g(x,y) &:= \begin{bmatrix} -y \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal value of the upper-level objective function is -29.2 and can be achieved at $(0.0, 0.9, 0.0, 0.6, 0.4)$, for example; cf. [23].

Problem name: `ClarkWesterberg1990a`

Source: [10]

Description: ClarkWesterberg1990a is defined as follows

$$\begin{aligned} F(x,y) &:= (x-3)^2+(y-2)^2 \\ G(x,y) &:= \begin{bmatrix} x-8 \\ -x \end{bmatrix} \\ f(x,y) &:= (y-5)^2 \\ g(x,y) &:= \begin{bmatrix} -2x+y-1 \\ x-2y+2 \\ x+2y-14 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(1.0, 3.0)$; cf. [47].

Problem name: `Colson2002BIPA1`

Source: [11]

Description: Colson2002BIPA1 is defined as follows

$$\begin{aligned} F(x,y) &:= (10-x)^3+(10-y)^3 \\ G(x,y) &:= \begin{bmatrix} x-5 \\ -x+y \\ -x \end{bmatrix} \\ f(x,y) &:= (x+2y-15)^4 \\ g(x,y) &:= \begin{bmatrix} x+y-20 \\ y-20 \\ -y \end{bmatrix} \end{aligned}$$

Comment: A global optimal solution is $(5, 5)$.

Problem name: `Colson2002BIPA2`

Source: [11]

Description: Colson2002BIPA2 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^2+(2y+1)^2 \\ G(x,y) &:= -x \\ f(x,y) &:= (y-1)^2-1.5xy+x^3 \\ g(x,y) &:= \begin{bmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known solution is $(1, 0)$; cf. [8].

Problem name: `Colson2002BIPA3`

Source: [11]

Description: Colson2002BIPA3 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^4+(2y+1)^4 \\ G(x,y) &:= \begin{bmatrix} x+y-4 \\ -x \end{bmatrix} \\ f(x,y) &:= \exp(-x+y)+x^2+2xy+y^2+2x+6y \\ g(x,y) &:= \begin{bmatrix} -x+y-2 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known solution is $(4, 0)$; cf. [8].

Problem name: `Colson2002BIPA4`

Source: [11]

Description: Colson2002BIPA4 as defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2 + (y-10)^2 \\ G(x,y) & := & \begin{bmatrix} x+2y-6 \\ -x \end{bmatrix} \\ f(x,y) & := & x^3 + 2y^3 + x - 2y - x^2 \\ g(x,y) & := & \begin{bmatrix} -x+2y-3 \\ -y \end{bmatrix} \end{array}$$

Comment: The best known solution is $(0, 0.6039)$; cf. [8].

Problem name: `Colson2002BIPA5`

Source: [11]

Description: Colson2002BIPA5 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-y_2)^4 + (y_1-1)^2 + (y_1-y_2)^2 \\ G(x,y) & := & -x \\ f(x,y) & := & 2x + \exp y_1 + y_1^2 + 4y_1 + 2y_2^2 - 6y_2 \\ g(x,y) & := & \begin{bmatrix} 6x + y_1^2 + \exp y_2 - 15 \\ 5x + y_1^4 - y_2 - 25 \\ -[4,2]^\top + y \\ -y \end{bmatrix} \end{array}$$

Comment: The best known solution is $(1.94, 0, 1.21)$; cf. [8].

Problem name: `Dempe1992a`

Source: [12]

Description: Dempe1992a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & y_2 \\ G(x,y) & := & x_1^2 + (x_2+1)^2 - 1 \\ f(x,y) & := & \frac{1}{2}(y_1-1)^2 + \frac{1}{2}y_2^2 \\ g(x,y) & := & \begin{bmatrix} y_1 + y_2 x_1 + x_2 \\ y_1 \end{bmatrix} \end{array}$$

Comment: One possible solution is $(0, 0, 0, -0.5)$.

Problem name: `Dempe1992b`

Source: [12]

Description: Dempe1992b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-3.5)^2 + (y+4)^2 \\ f(x,y) & := & (y-3)^2 \\ g(x,y) & := & y^2 - x \end{array}$$

Comment: The global upper-and lower-level optimal values are respectively obtained as 31.25 and 4.00 in the paper [8].

Problem name: `DempeDutta2012Ex24`

Source: [14]

Description: DempeDutta2012Ex24 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-1)^2 + y^2 \\ f(x,y) &:= x^2 y \\ g(x,y) &:= y^2 \end{aligned}$$

Comment: The global optimal solution of the problem is $(1,0)$; cf. [14].

Problem name: `DempeDutta2012Ex31`

Source: [14]

Description: DempeDutta2012Ex31 is defined as follows

$$\begin{aligned} F(x,y) &:= -y_2 \\ G(x,y) &:= \begin{bmatrix} -x \\ y_1 y_2 \\ -y_1 y_2 \end{bmatrix} \\ f(x,y) &:= y_1^2 + (y_2+1)^2 \\ g(x,y) &:= \begin{bmatrix} (y_1-x_1)^2 + (y_2-x_1-1)^2 - 1 \\ (y_1+x_2)^2 + (y_2-x_2-1)^2 - 1 \end{bmatrix} \end{aligned}$$

Comment: The point $(0.71, 0.71, 0, 1)$ is global optimal solution of the problem provided in [14, 42].

Problem name: `DempeEtal2012`

Source: [15]

Description: DempeEtal2012 is defined as follows

$$\begin{aligned} F(x,y) &:= x \\ G(x,y) &:= \begin{bmatrix} -1-x \\ x-1 \end{bmatrix} \\ f(x,y) &:= xy \\ g(x,y) &:= \begin{bmatrix} -y \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution is $(-1,1)$; cf. [15].

Problem name: `DempeFranke2011Ex41`

Source: [16]

Description: DempeFranke2011Ex41 is defined as follows

$$\begin{aligned} F(x,y) &:= x_1 + y_1^2 + y_2^2 \\ G(x,y) &:= \begin{bmatrix} -1-x_1 \\ -1+x_1 \\ -1-x_2 \\ 1+x_2 \end{bmatrix} \\ f(x,y) &:= x^\top y \\ g(x,y) &:= \begin{bmatrix} -2y_1+y_2 \\ y_1-2 \\ -y_2 \\ y_2-2 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution is $(0,-1,1,2)$; cf. [16].

Problem name: `DempeFranke2011Ex42`

Source: [16]

Description: DempeFranke2011Ex42 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x_1 + (y_1-1)^2 + y_2^2 \\
G(x,y) &:= \begin{bmatrix} -1-x_1 \\ -1+x_1 \\ -1-x_2 \\ 1+x_2 \end{bmatrix} \\
f(x,y) &:= x^\top y \\
g(x,y) &:= \begin{bmatrix} -y_1+y_2-1 \\ y_1+y_2-3.5 \\ y_2-2 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(1,-1,0,1)$; cf. [16].

Problem name: `DempeFranke2014Ex38`

Source: [17]

Description: DempeFranke2014Ex38 is defined as follows

$$
\begin{aligned}
F(x,y) &:= 2x_1 + x_2 + 2y_1 - y_2 \\
G(x,y) &:= \begin{bmatrix} -1-x_1 \\ -1+x_1 \\ -1-x_2 \\ x_2+0.75 \end{bmatrix} \\
f(x,y) &:= x^\top y \\
g(x,y) &:= \begin{bmatrix} -2y_1+y_2 \\ y_1-2 \\ -y_2 \\ y_2-2 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(-1,-1,2,2)$; cf. [17].

Problem name: `DempeLohse2011Ex31a`

Source: [18]

Description: DempeLohse2011Ex31a is defined as follows

$$
\begin{aligned}
F(x,y) &:= (x_1-0.5)^2 + (x_2-0.5)^2 - 3y_1 - 3y_2 \\
f(x,y) &:= x_1y_1 + x_2y_2 \\
g(x,y) &:= \begin{bmatrix} y_1+y_2-2 \\ -y_1+y_2 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: Authors in [18] claimed the point $(0.5,0.5,1,1)$ is the unique global optimal solution to this problem, which actually is not correct since solution set of the lower level problem is $\{(0,0)\}$ when $x=(0.5,0.5)$; The true unique global solution should be $(0,0,1,1)$.

Problem name: `DempeLohse2011Ex31b`

Source: [18]

Description: DempeLohse2011Ex31b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x_1-0.5)^2+(x_2-0.5)^2+x_3^2-3y_1-3y_2-6y_3 \\ f(x,y) & := & x_1y_1+x_2y_2+x_3y_3 \\ g(x,y) & := & \begin{bmatrix} y_1+y_2+y_3-2 \\ -y_1+y_2 \\ -y \end{bmatrix} \end{array}$$

Comment: The point $(0.5, 0.5, 0, 1, 1, 0)$ is a local optimal solution of the problem according to [18]. While a suggested global optimal solution is $(0.5, 0.5, 0, 0, 0, 2)$.

Problem name: `DeSilva1978`

Source: [19]

Description: DeSilva1978 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1^2-2x_1+x_2^2-2x_2+y_1^2+y_2^2 \\ f(x,y) & := & (y_1-x_1)^2+(y_2-x_2)^2 \\ g(x,y) & := & \begin{bmatrix} -y+(0.5)_2 \\ y-(1.5)_2 \end{bmatrix} \end{array}$$

Comment: The global optimal upper-and lower-level values obtained in [8] are -1.00 and 0.00, respectively. Corresponding global optimal solution is $(0.5, 0.5, 0.5, 0.5)$.

Problem name: `FalkLiu1995`

Source: [20]

Description: FalkLiu1995 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1^2-3x_1+x_2^2-3x_2+y_1^2+y_2^2 \\ f(x,y) & := & (y_1-x_1)^2+(y_2-x_2)^2 \\ g(x,y) & := & \begin{bmatrix} -y+(0.5)_2 \\ y-(1.5)_2 \end{bmatrix} \end{array}$$

Comment: The optimal solution is $(\sqrt{3}/2, \sqrt{3}/2, \sqrt{3}/2, \sqrt{3}/2)$ according to [20].

Problem name: `FloudasEtal2013`

Source: [21]

Description: FloudasEtal2013 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & 2x_1+2x_2-3y_1-3y_2-60 \\ G(x,y) & := & \begin{bmatrix} -x \\ x-50_2 \end{bmatrix} \\ f(x,y) & := & (y_1-x_1+20)^2+(y_2-x_2+20)^2 \\ g(x,y) & := & \begin{bmatrix} x_1+x_2+y_1-2y_2-40 \\ 2y-x+10_2 \\ -y-10_2 \\ y-20_2 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(0, 0, -10, -10)$; [52].

Problem name: `FloudasZlobec1998`

Source: [22]

Description: FloudasZlobec1998 is defined as follows

$$\begin{aligned}
F(x,y) &:= x^3 y_1 + y_2\\
G(x,y) &:= \begin{bmatrix} x-1 \\ -x \end{bmatrix}\\
f(x,y) &:= -y_2\\
g(x,y) &:= \begin{bmatrix} -y_1 - 1 \\ y_1 - 1 \\ -y_2 \\ y_2 - 100 \\ xy_1 - 10 \\ y_1^2 + xy_2 - 1 \end{bmatrix}
\end{aligned}$$

Comment: Notice that explicit bounds on the variable y were added. This is same as [39]. The global optimal solution is $(1, 0, 1)$ according to [23, 39].

Problem name: `GumusFloudas2001Ex1`

Source: [23]

Description: GumusFloudas2001Ex1 is defined as follows

$$\begin{aligned}
F(x,y) &:= 16x^2 + 9y^2\\
G(x,y) &:= \begin{bmatrix} -x \\ x - 12.5 \\ -4x + y \end{bmatrix}\\
f(x,y) &:= (x + y - 20)^4\\
g(x,y) &:= \begin{bmatrix} -y \\ y - 50 \\ 4x + y - 50 \end{bmatrix}
\end{aligned}$$

Comment: The global optimal solution of the problem is $(11.25, 5)$; cf. [39].

Problem name: `GumusFloudas2001Ex3`

Source: [23]

Description: GumusFloudas2001Ex3 is defined as follows

$$\begin{aligned}
F(x,y) &:= -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3\\
G(x,y) &:= \begin{bmatrix} -x \\ x - 2_2 \end{bmatrix}\\
f(x,y) &:= \frac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3}\\
g(x,y) &:= \begin{bmatrix} -y \\ y - 2_3 \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{bmatrix}
\end{aligned}$$

Comment: The global optimal solution of the problem is $(0, 0.9, 0, 0.6, 0.4)$; cf. [39].

Problem name: `GumusFloudas2001Ex4`

Source: [23]

Description: GumusFloudas2001Ex4 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-3)^2+(y-2)^2 \\ G(x,y) &:= \begin{bmatrix} -x \\ x-8 \\ -2x+y-1 \\ x-2y+2 \\ x+2y-14 \end{bmatrix} \\ f(x,y) &:= (y-5)^2 \\ g(x,y) &:= \begin{bmatrix} -y \\ y-10 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(3,5)$; cf. [39].

Problem name: `GumusFloudas2001Ex5`

Source: [23]

Description: GumusFloudas2001Ex5 is defined as follows

$$\begin{aligned} F(x,y) &:= x \\ G(x,y) &:= \begin{bmatrix} -x+0.1 \\ x-10 \end{bmatrix} \\ f(x,y) &:= -y_1+0.5864y_1^{0.67} \\ g(x,y) &:= \begin{bmatrix} -y+(0.1)_2 \\ y-10_2 \\ \frac{0.0332333}{y_2}+0.1y_1-1 \\ 4\frac{x}{y_2}+2\frac{x^{-0.71}}{y_2}+0.0332333x^{-1.3}-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution is $(0.193616, 9.9667667, 10)$; cf. [39].

Problem name: `HatzEtal2013`

Source: [24]

Description: HatzEtal2013 is defined as follows

$$\begin{aligned} F(x,y) &:= -x+2y_1+y_2 \\ f(x,y) &:= (x-y_1)^2+y_2^2 \\ g(x,y) &:= -y \end{aligned}$$

Comment: The global optimal solution of the problem is $(0,0,0)$; cf. [24].

Problem name: `HendersonQuandt1958`

Source: [25]

Description: HendersonQuandt1958 is defined as follows

$$\begin{aligned} F(x,y) &:= \tfrac{1}{2}x^2+\tfrac{1}{2}xy-95x \\ G(x,y) &:= \begin{bmatrix} x-200 \\ -x \end{bmatrix} \\ f(x,y) &:= y^2+(\tfrac{1}{2}x-100)y \\ g(x,y) &:= -y \end{aligned}$$

Comment: The best known solution from [27] is $(93.33333, 26.667)$.

Problem name: `HenrionSurowiec2011`

Source: [26]

Description: HenrionSurowiec2011 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2 + cy \\ f(x,y) & := & 0.5y^2 - xy \end{array}$$

Comment: Here, c is a real-valued parameter. The global optimal solution of the problem is $-0.5c(1,1)$; cf. [26].

Problem name: `IshizukaAiyoshi1992a`

Source: [28]

Description: IshizukaAiyoshi1992a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & xy_2^2 \\ G(x,y) & := & -x - M \\ f(x,y) & := & y_1 \\ g(x,y) & := & \begin{bmatrix} -x \\ -x - y_1 \\ y_1 - x \\ -M - y_1 - y_2 \\ y_1 + y_2 - M \end{bmatrix} \end{array}$$

Comment: Here, M is assumed to be an arbitrarily large number such that $M > 1$. From [28], $(x^*, -M, 0)$ is the global optimal solution where $x^* \in [0, M]$.

Problem name: `KleniatiAdjiman2014Ex3`

Source: [29]

Description: KleniatiAdjiman2014Ex3 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x - y \\ G(x,y) & := & \begin{bmatrix} -x - 1 \\ x - 1 \end{bmatrix} \\ f(x,y) & := & 0.5xy^2 - xy^3 \\ g(x,y) & := & \begin{bmatrix} -y - 1 \\ y - 1 \end{bmatrix} \end{array}$$

Comment: $(0,1)$ is the global optimal solution of the problem according to [29, 47].

Problem name: `KleniatiAdjiman2014Ex4`

Source: [29]

Description: KleniatiAdjiman2014Ex4 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -\sum_{j=1}^{5}\left(x_j^2 + y_j^2\right) \\ G(x,y) & := & \begin{bmatrix} y_1y_2 - x_1 \\ x_2y_1^2 \\ x_1 - \exp x_2 + y_3 \\ -x - 1_5 \\ x - 1_5 \end{bmatrix} \\ f(x,y) & := & y_1^3 + y_2^2x_1 + y_2^2x_2 + 0.1y_3 + (y_4^2 + y_5^2)x_3x_4x_5 \\ g(x,y) & := & \begin{bmatrix} x_1 - y_3^2 - 0.2 \\ -y - 1_5 \\ y - 1_5 \end{bmatrix} \end{array}$$

Comment: $(1.0, -(1.0)_9)$ is the best known solution of the problem according to [29, 47].

Problem name: `LamparielloSagratella2017Ex23`

Source: [30]

Description: LamparielloSagratella2017Ex23 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) & := & (x-y_1)^2+(y_2+1)^2 \\ g(x,y) & := & \begin{bmatrix} y_1^3-y_2 \\ -y_2 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(-1,-1,0)$; cf. [30].

Problem name: `LamparielloSagratella2017Ex31`

Source: [31]

Description: LamparielloSagratella2017Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & -x+1 \\ f(x,y) & := & y \\ g(x,y) & := & -x-y+1 \end{array}$$

Comment: The global optimal solution is $(1,0)$; cf. [31].

Problem name: `LamparielloSagratella2017Ex32`

Source: [31]

Description: LamparielloSagratella2017Ex32 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ f(x,y) & := & (x+y-1)^2 \end{array}$$

Comment: The global optimal solution is $(0.5,0.5)$; cf. [31].

Problem name: `LamparielloSagratella2017Ex33`

Source: [31]

Description: LamparielloSagratella2017Ex33 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+(y_1+y_2)^2 \\ G(x,y) & := & -x+0.5 \\ f(x,y) & := & y_1 \\ g(x,y) & := & \begin{bmatrix} -x-y_1-y_2+1 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(0.5,0,0.5)$; cf. [31].

Problem name: `LamparielloSagratella2017Ex35`

Source: [31]

Description: LampariellоSagratella2017Ex35 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & x^2+y^2 \\
G(x,y) & := & \begin{bmatrix} -1-x \\ x-1 \end{bmatrix} \\
f(x,y) & := & -y \\
g(x,y) & := & \begin{bmatrix} 2x+y-2 \\ -y \\ y-1 \end{bmatrix}
\end{array}
$$

Comment: The global optimal solution is $(\frac{4}{5}, \frac{2}{5})$; cf. [31].

Problem name: `LucchettiEtal1987`

Source: [33]

Description: LucchettiEtal1987 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & 0.5(1-x)+xy \\
G(x,y) & := & \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\
f(x,y) & := & (x-1)y \\
g(x,y) & := & \begin{bmatrix} -y \\ y-1 \end{bmatrix}
\end{array}
$$

Comment: The global optimal solution of the problem is $(1, 0)$; cf. [33].

Problem name: `LuDebSinha2016a`

Source: [34]

Description: LuDebSinha2016a is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & 2-\exp\left[-\left(\frac{0.2y-x+0.6}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{0.15y-0.4+x}{0.3}\right)^{2}\right] \\
G(x,y) & := & \begin{bmatrix} -x \\ x-1 \\ -y \\ y-2 \end{bmatrix} \\
f(x,y) & := & 2-\exp\left[-\left(\frac{1.5y-x}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{2y-3+x}{0.5}\right)^{2}\right]
\end{array}
$$

Comment: $(0.2, 1.4)$ is the best known solution for the problem; [34].

Problem name: `LuDebSinha2016b`

Source: [34]

Description: LuDebSinha2016b is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (x-0.5)^2+(y-1)^2 \\
G(x,y) & := & \begin{bmatrix} -x \\ x-1 \\ -y \\ y-2 \end{bmatrix} \\
f(x,y) & := & 2-\exp\left[-\left(\frac{1.5y-x}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{2y-3+x}{0.5}\right)^{2}\right]
\end{array}
$$

Comment: $(0.5, 1)$ is the best known solution for the problem; [34].

Problem name: `LuDebSinha2016c`

Source: [34]

Description: LuDebSinha2016c is defined as follows

$$
\begin{aligned}
F(x,y) &:= 2 - \exp\left[-\left(\tfrac{0.2y-x+0.6}{0.055}\right)^{0.4}\right] - 0.8\exp\left[-\left(\tfrac{0.15y-0.4+x}{0.3}\right)^{2}\right]\\
G(x,y) &:= \begin{bmatrix} -x \\ x-1 \\ -y \\ y-2 \end{bmatrix}\\
f(x,y) &:= (x-0.5)^2 + (y-1)^2
\end{aligned}
$$

Comment: $(0.26, 1)$ is the best known possible solution for the problem; [34].

Problem name: `LuDebSinha2016d`

Source: [34]

Description: LuDebSinha2016d is defined as follows

$$
\begin{aligned}
F(x,y) &:= -x_2\\
g(x,y) &:= \begin{bmatrix} -\left(\frac{y_1}{14}+\frac{16}{7}\right)(x_1-2)^2 + x_2 \\ -x_2 + \left(\frac{y_1}{14}+\frac{16}{7}\right)(x_1-5) \\ -\left[x_1+4-\left(\frac{y_1}{14}+\frac{16}{7}\right)\right]\left[x_1+8-\left(\frac{y_1}{14}+\frac{16}{7}\right)\right]+x_2 \\ -4-x_1 \\ -10+x_1 \\ -100-x_2 \\ -200+x_2 \\ -4-y_1 \\ -10+y_1 \\ -100-y_2 \\ -200+y_2 \end{bmatrix}\\
f(x,y) &:= -y_2\\
g(x,y) &:= \begin{bmatrix} -\left(\frac{x_1}{14}+\frac{16}{7}\right)(y_1-2)^2 + y_2 \\ -y_2 + 12.5\left(\frac{x_1}{14}+\frac{16}{7}\right)(y_1-5) \\ -5\left[y_1+4-\left(\frac{x_1}{14}+\frac{16}{7}\right)\right]\left[y_1+8-\left(\frac{x_1}{14}+\frac{16}{7}\right)\right]+y_2 \end{bmatrix}
\end{aligned}
$$

Comment: Solutions are unknown. A possible solution is $(10, 192, 10, 192)$.

Problem name: `LuDebSinha2016e`

Source: [34]

Description: LuDebSinha2016e is defined as follows

$$\begin{aligned}
F(x,y) &:= \left(\tfrac{y_2-50}{30}\right)^2 + \left(\tfrac{x-2.5}{0.2}\right)^2\\
G(x,y) &:= \begin{bmatrix} 2-x \\ -3+x \\ -4-y_1 \\ -10+y_1 \\ -100-y_2 \\ -200+y_2 \end{bmatrix}\\
f(x,y) &:= -y_2\\
g(x,y) &:= \begin{bmatrix} -x(y_1-2)^2+y_2 \\ -y_2+12.5x(y_1-5) \\ -5(y_1+4-x)(y_1+8-x)+y_2 \end{bmatrix}
\end{aligned}$$

Comment: Solutions are unknown.

Problem name: `LuDebSinha2016f`

Source: [34]

Description: LuDebSinha2016f is defined as follows

$$\begin{aligned}
F(x,y) &:= -x_2\\
G(x,y) &:= \begin{bmatrix} 2-y \\ -4+y \\ -80-x_1 \\ -200+x_1 \\ -100-x_2 \\ -200+x_2 \\ -y\left(\tfrac{x_1}{20}-2\right)+x_2 \\ -x_2+12.5y\left(\tfrac{x_1}{20}-5\right) \\ -5\left(\tfrac{x_1}{20}+4-y\right)\left(\tfrac{x_1}{20}+8-y\right)+x_2 \end{bmatrix}\\
f(x,y) &:= \left(\tfrac{x_1-50}{28}\right)^2 + \left(\tfrac{y-2.5}{0.2}\right)^2
\end{aligned}$$

Comment: Solutions are unknown.

Problem name: `MacalHurter1997`

Source: [36]

Description: MacalHurter1997 is defined as follows

$$\begin{aligned}
F(x,y) &:= (x-1)^2+(y-1)^2\\
f(x,y) &:= 0.5y^2+500y-50xy
\end{aligned}$$

Comment: The global optimal solution is $(10.0163, 0.8197)$; cf. [36].

Problem name: `Mirrlees1999`

Source: [38]

Description: Mirrlees1999 is defined as follows

$$\begin{aligned}
F(x,y) &:= (x-2)^2+(y-1)^2\\
f(x,y) &:= -x\exp\left[-(y+1)^2\right]-\exp\left[-(y-1)^2\right]\\
g(x,y) &:= \begin{bmatrix} -2-y \\ -2+y \end{bmatrix}
\end{aligned}$$

Comment: We used the version from [55], which added the box constriants on y but the global optimal remains the same. This problem is known in the literature as Mirrlees problem. It is usually used to illustrate how the KKT reformulation of the bilevel optimization problem is not appropriate for problems with nonconvex lower-level problems. The global optimal solution for the problem is $(1, 0.95753)$; [38].

Problem name: `MitsosBarton2006Ex38`

Source: [39]

Description: MitsosBarton2006Ex38 is defined as follows

$$\begin{aligned} F(x,y) &:= y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \\ -y-0.1 \\ y-0.1 \end{bmatrix} \\ f(x,y) &:= (x+\exp x)y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(-0.567, 0)$; cf. [39].

Problem name: `MitsosBarton2006Ex39`

Source: [39]

Description: MitsosBarton2006Ex39 is defined as follows

$$\begin{aligned} F(x,y) &:= x \\ G(x,y) &:= \begin{bmatrix} -x+y \\ -x+10 \\ x-10 \end{bmatrix} \\ f(x,y) &:= y^3 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(-1, -1)$; cf. [39].

Problem name: `MitsosBarton2006Ex310`

Source: [39]

Description: MitsosBarton2006Ex310 is defined as follows

$$\begin{aligned} F(x,y) &:= y \\ G(x,y) &:= \begin{bmatrix} -x+0.1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= x(16y^4+2y^3-8y^2-1.5y+0.5) \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The set of all global optimal solution is $[0.1,\ 1] \times \{0.5\}$; cf. [39].

Problem name: `MitsosBarton2006Ex311`

Source: [39]

Description: MitsosBarton2006Ex311 is defined as follows

$$\begin{aligned} F(x,y) &:= y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= x(16y^4+2y^3-8y^2-1.5y+0.5) \\ g(x,y) &:= \begin{bmatrix} -y-0.8 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0,-0.8)$; cf. [39].

Problem name: `MitsosBarton2006Ex312`

Source: [39]

Description: MitsosBarton2006Ex312 is defined as follows

$$\begin{aligned} F(x,y) &:= -x+xy+10y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= -xy^2+0.5y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0,0)$; cf. [39].

Problem name: `MitsosBarton2006Ex313`

Source: [39]

Description: MitsosBarton2006Ex313 is defined as follows

$$\begin{aligned} F(x,y) &:= x-y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= 0.5xy^2-x^3y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0,1)$; cf. [39].

Problem name: `MitsosBarton2006Ex314`

Source: [39]

Description: MitsosBarton2006Ex314 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-0.25)^2+y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= \tfrac{1}{3}y^3-xy \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0.25,0.5)$; cf. [39].

Problem name: `MitsosBarton2006Ex315`

Source: [39]

Description: MitsosBarton2006Ex315 is defined as follows

$$\begin{aligned} F(x,y) &:= x+y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= \tfrac{1}{2}xy^2 - \tfrac{1}{3}y^3 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(-1,1)$; cf. [39].

Problem name: `MitsosBarton2006Ex316`

Source: [39]

Description: MitsosBarton2006Ex316 is defined as follows

$$\begin{aligned} F(x,y) &:= 2x+y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= -\tfrac{1}{2}xy^2 - \tfrac{1}{4}y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The points $(-1,0)$ and $(-0.5,-1)$ are the two global optimal solutions of the problem; cf. [39].

Problem name: `MitsosBarton2006Ex317`

Source: [39]

Description: MitsosBarton2006Ex317 is defined as follows

$$\begin{aligned} F(x,y) &:= (x+\tfrac{1}{2})^2 + \tfrac{1}{2}y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= \tfrac{1}{2}xy^2 + \tfrac{1}{4}y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The points $(-0.25,0.5)$ and $(-0.25,-0.5)$ are the two global optimal solutions of the problem; cf. [39].

Problem name: `MitsosBarton2006Ex318`

Source: [39]

Description: MitsosBarton2006Ex318 is defined as follows

$$\begin{aligned} F(x,y) &:= -x^2+y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= xy^2 - \tfrac{1}{2}y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0.5,0)$; cf. [39].

Problem name: `MitsosBarton06Ex319`

Source: [39]
Description: MitsosBarton2006Ex319 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & xy - y + \frac{1}{2}y^2 \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) & := & -xy^2 + \frac{1}{2}y^4 \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(0.189, 0.4343)$; cf. [39].

Problem name: `MitsosBarton2006Ex320`
Source: [39]
Description: MitsosBarton2006Ex320 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x - \frac{1}{4})^2 + y^2 \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) & := & \frac{1}{3}y^3 - x^2y \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(0.5, 0.5)$; cf. [39].

Problem name: `MitsosBarton2006Ex321`
Source: [39]
Description: MitsosBarton2006Ex321 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x+0.6)^2 + y^2 \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) & := & y^4 + \frac{4}{30}(-x+1)y^3 + (-0.02x^2 + 0.16x - 0.4)y^2 \\ & & \qquad + (0.004x^3 - 0.036x^2 + 0.08x)y \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(-0.5545, 0.4554)$; cf. [39].

Problem name: `MitsosBarton2006Ex322`
Source: [39]
Description: MitsosBarton2006Ex322 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x+0.6)^2 + y^2 \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) & := & y^4 + \frac{2}{15}(-x+1)y^3 + (-0.02x^2 + 0.16x - 0.4)y^2 \\ & & \qquad + (0.004x^3 - 0.036x^2 + 0.08x)y \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \\ 0.01(1+x)^2 - y^2 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(-0.5545, 0.4554)$; cf. [39].

Problem name: `MitsosBarton2006Ex323`

Source: [39]

Description: MitsosBarton2006Ex323 is defined as follows

$$\begin{aligned}
F(x,y) &:= x^2 \\
G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \\ 1+x-9x^2-y \end{bmatrix} \\
f(x,y) &:= y \\
g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \\ y^2(x-0.5) \end{bmatrix}
\end{aligned}$$

Comment: The global optimal solution of the problem is $(-0.4191, -1)$; cf. [39].

Problem name: `MitsosBarton2006Ex324`

Source: [39]

Description: MitsosBarton2006Ex324 is defined as follows

$$\begin{aligned}
F(x,y) &:= x^2 - y \\
G(x,y) &:= \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\
f(x,y) &:= \left[(y-1-0.1x)^2 - 0.5 - 0.5x\right]^2 \\
g(x,y) &:= \begin{bmatrix} -y \\ y-3 \end{bmatrix}
\end{aligned}$$

Comment: The global optimal solution of the problem is $(0.2106, 1.799)$; cf. [39].

Problem name: `MitsosBarton2006Ex325`

Source: [39]

Description: MitsosBarton2006Ex325 is defined as follows

$$\begin{aligned}
F(x,y) &:= x_1y_1 + x_2y_1^2 - x_1x_2y_3 \\
G(x,y) &:= \begin{bmatrix} -x-1_2 \\ x-1_2 \\ 0.1y_1y_2 - x_1^2 \\ x_2y_1^2 \end{bmatrix} \\
f(x,y) &:= x_1y_1^2 + x_2y_2y_3 \\
g(x,y) &:= \begin{bmatrix} -y-1_3 \\ y-1_3 \\ y_1^2 - y_2y_3 \\ y_2^2y_3 - y_1x_1 \\ -y_3^2 + 0.1 \end{bmatrix}
\end{aligned}$$

Comment: The best known value for the upper-level objective function is -1 and a corresponding point is $(-1, -1, -1, 1, 1)$; cf. [39].

Problem name: `MitsosBarton2006Ex326`

Source: [39]

Description: MitsosBarton2006Ex326 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1y_1 + x_2y_2^2 + x_1x_2y_3^3 \\ G(x,y) & := & \begin{bmatrix} 0.1 - x_1^2 \\ 1.5 - y_1^2 - y_2^2 - y_3^2 \\ 2.5 + y_1^2 + y_2^2 + y_3^2 \\ -x - 1_2 \\ x - 1_2 \end{bmatrix} \\ f(x,y) & := & x_1y_1^2 + x_2y_2^2 + (x_1 - x_2)y_3^2 \\ g(x,y) & := & \begin{bmatrix} -y - 1_3 \\ y - 1_3 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(-1, -1, 1, 1, -0.707)$; cf. [39].

Problem name: `MitsosBarton2006Ex327`

Source: [39]

Description: MitsosBarton2006Ex327 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \sum_{j=1}^{5}(x_j^2 + y_j^2) \\ G(x,y) & := & \begin{bmatrix} -x - 1_5 \\ x - 1_5 \\ y_1y_2 - x_1 \\ x_2y_1^2 \\ x_1 - \exp x_2 + y_3 \end{bmatrix} \\ f(x,y) & := & y_1^3 + y_2^2x_1 + y_2^2x_2 + 0.1y_3 + (y_4^2 + y_5^2)x_3x_4x_5 \\ g(x,y) & := & \begin{bmatrix} -y - 1_5 \\ y - 1_5 \\ y_1y_2 - 0.3 \\ x_1 - y_3^2 - 0.2 \\ -\exp y_3 + y_4y_5 - 0.1 \end{bmatrix} \end{array}$$

Comment: The best known value for the upper-level objective function is 2 and a corresponding point is $(0_5, -1, 0, -1, 0, 0)$; cf. [39].

Problem name: `MitsosBarton2006Ex328`

Source: [39]

Description: MitsosBarton2006Ex328 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -\sum_{j=1}^{5}(x_j^2 + y_j^2) \\ G(x,y) & := & \begin{bmatrix} -x - 1_5 \\ x - 1_5 \\ y_1y_2 - x_1 \\ x_2y_1^2 \\ x_1 - \exp x_2 + y_3 \end{bmatrix} \\ f(x,y) & := & y_1^3 + y_2^2x_1 + y_2^2x_2 + 0.1y_3 + (y_4^2 + y_5^2)x_3x_4x_5 \\ g(x,y) & := & \begin{bmatrix} -y - 1_5 \\ y - 1_5 \\ y_1y_2 - 0.3 \\ x_1 - y_3^2 - 0.2 \\ -\exp y_3 + y_4y_5 - 0.1 \end{bmatrix} \end{array}$$

Comment: The best known values for the upper and lower-level objective functions are -10 and -3.1 respectively, and a corresponding point is $(1,(-1)_5,1,-1,-1,1)$; cf. [39]. Another possible solution is $((-1)_5,1,-1,-1,-1,1)$ with the same the upper and lower-level objective function values.

Problem name: `MorganPatrone2006a`

Source: [40]

Description: MorganPatrone2006a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(x+y) \\ G(x,y) & := & \begin{bmatrix} -x-0.5 \\ x-0.5 \end{bmatrix} \\ f(x,y) & := & xy \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(0,1)$; cf. [40].

Problem name: `MorganPatrone2006b`

Source: [40]

Description: MorganPatrone2006b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(x+y) \\ f(x,y) & := & \begin{cases} (x+0.25)y & \text{if } x \in [-0.5,-0.25] \\ 0 & \text{if } x \in [-0.25,0.25] \\ (x-0.25)y & \text{if } x \in [0.25,0.5] \end{cases} \\ g(x,y) & := & \begin{bmatrix} -0.5-x \\ -0.5+x \\ -1-y \\ -1+y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(0.25,1)$; cf. [40].

Problem name: `MorganPatrone2006c`

Source: [40]

Description: MorganPatrone2006c is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(x+y) \\ f(x,y) & := & \begin{cases} \left[x+-\frac{7}{4}\right]y & \text{if } x \in \left[-2,-\frac{7}{4}\right] \\ 0 & \text{if } x \in \left[-\frac{7}{4},\frac{7}{4}\right] \\ \left[x--\frac{7}{4}\right]y & \text{if } x \in \left[\frac{7}{4},2\right] \end{cases} \\ g(x,y) & := & \begin{bmatrix} -x-2 \\ x-2 \\ -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(2,-1)$; cf. [40].

Problem name: `MuuQuy2003Ex1`

Source: [41]

Description: MuuQuy2003Ex1 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & y_1^2 + y_2^2 + x^2 - 4x \\ G(x,y) & := & \begin{bmatrix} -x \\ x-2 \end{bmatrix} \\ f(x,y) & := & y_1^2 + \frac{1}{2}y_2^2 + y_1y_2 + (1-3x)y_1 + (1+x)y_2 \\ g(x,y) & := & \begin{bmatrix} 2y_1 + y_2 - 2x - 1 \\ -y \end{bmatrix} \end{array}$$

Comment: The best known solution is $(0.8438, 0.7657, 0)$ according to [41].

Problem name: `MuuQuy2003Ex2`

Source: [41]

Description: MuuQuy2003Ex2 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & y_1^2 + y_3^2 - y_1y_3 - 4y_2 - 7x_1 + 4x_2 \\ G(x,y) & := & \begin{bmatrix} -x \\ x_1 + x_2 - 1 \end{bmatrix} \\ f(x,y) & := & y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2 + y_1y_2 + (1-3x_1)y_1 + (1+x_2)y_2 \\ g(x,y) & := & \begin{bmatrix} 2y_1 + y_2 - y_3 + x_1 - 2x_2 + 2 \\ -y \end{bmatrix} \end{array}$$

Comment: The best known solution is $(0.609, 0.391, 0.000, 0.000, 1.828)$; cf. [41].

Problem name: `NieEtal2017Ex34`

Source: [42]

Description: NieEtal2017Ex34 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x + y_1 + y_2 \\ G(x,y) & := & \begin{bmatrix} -x+2 \\ x-3 \end{bmatrix} \\ f(x,y) & := & x(y_1 + y_2) \\ g(x,y) & := & \begin{bmatrix} -y_1^2 + y_2^2 + (y_1^2 + y_2^2)^2 \\ -y_1 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(2, 0, 0)$; cf. [42].

Problem name: `NieEtal2017Ex52`

Source: [42]

Description: NieEtal2017Ex52 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1y_1 + x_2y_2 + x_1x_2y_1y_2y_3 \\ G(x,y) & := & \begin{bmatrix} -x - 1_2 \\ x - 1_2 \\ y_1y_2 - x_1^2 \end{bmatrix} \\ f(x,y) & := & x_1y_1^2 + x_2^2y_2y_3 - y_1y_3^2 \\ g(x,y) & := & \begin{bmatrix} 1 - y_1^2 - y_2^2 - y_3^2 \\ y_1^2 + y_2^2 + y_3^2 - 2 \end{bmatrix} \end{array}$$

Comment: The point $(-1, -1, 1.1097, 0.3143, -0.8184)$ is global optimal solution of the problem provided in [42].

Problem name: `NieEtal2017Ex54`

Source: [42]
Description: NieEtal2017Ex54 is defined as follows

$$\begin{aligned}
F(x,y) &:= x_1^2y_1 + x_2y_2 + x_3y_3^2 + x_4y_4^2 \\
G(x,y) &:= \begin{bmatrix} \|x\|^2 - 1 \\ y_1y_2 - x_1 \\ y_3y_4 - x_3^2 \end{bmatrix} \\
f(x,y) &:= y_1^2 - y_2(x_1+x_2) - (y_3+y_4)(x_3+x_4) \\
g(x,y) &:= \begin{bmatrix} \|y\|^2 - 1 \\ y_2^2 + y_3^2 + y_4^2 - y_1 \end{bmatrix}
\end{aligned}$$

Comment: $(0, -0, -0.7071, -0.7071, 0.6180, 0, -0.5559, -0.5559)$ is the global optimal solution of the problem obtained in [42].

Problem name: `NieEtal2017Ex57`
Source: [42]
Description: NieEtal2017Ex57 is defined as follows

$$\begin{aligned}
F(x,y) &:= \tfrac{1}{2}x_1^2y_1 + x_2y_2^2 - (x_1 + x_2^2)y_3 \\
G(x,y) &:= \begin{bmatrix} -x - 1_2 \\ x - 1_2 \\ -x_1 - x_2 + x_1^2 + y_1^2 + y_2^2 \end{bmatrix} \\
f(x,y) &:= x_2(y_1y_2y_3 + y_2^2 - y_3^3) \\
g(x,y) &:= \begin{bmatrix} -x_1 + y_1^2 + y_2^2 + y_3^2 \\ -1 + 2y_2y_3 \end{bmatrix}
\end{aligned}$$

Comment: The point $(1, 1, 0, 0, 1)$ is the best known solution of the problem provided in [42].

Problem name: `NieEtal2017Ex58`
Source: [42]
Description: NieEtal2017Ex58 is defined as follows

$$\begin{aligned}
F(x,y) &:= (x_1+x_2+x_3+x_4)(y_1+y_2+y_3+y_4) \\
G(x,y) &:= \begin{bmatrix} \|x\|^2 - 1 \\ y_3^2 - x_4 \\ y_2y_4 - x_1 \end{bmatrix} \\
f(x,y) &:= x_1y_1 + x_2y_2 + 0.1y_3 + 0.5y_4 - y_3y_4 \\
g(x,y) &:= \begin{bmatrix} y_1^2 + 2y_2^2 + 3y_3^2 + 4y_4^2 - x_1^2 - x_3^2 - x_2 - x_4 \\ -y_2y_3 + y_1y_4 \end{bmatrix}
\end{aligned}$$

Comment: $(0.5135, 0.5050, 0.4882, 0.4929, -0.8346, -0.4104, -0.2106, -0.2887)$ is the best known solution of the problem obtained in [42].

Problem name: `NieEtal2017Ex61`
Source: [42]

Description: NieEtal2017Ex61 is defined as follows

$$
\begin{aligned}
F(x,y) &:= y_1^3(x_1^2 - 3x_1x_2) - y_1^2 y_2 + y_2 x_2^3 \\
G(x,y) &:= \begin{bmatrix} -x - 1_2 \\ x - 1_2 \\ -y_2 - y_1(1 - x_1^2) \end{bmatrix} \\
f(x,y) &:= y_1 y_2^2 - y_2^3 - y_1^2(x_2 - x_1^2) \\
g(x,y) &:= y_1^2 + y_2^2 - 1
\end{aligned}
$$

Comment: The point $(0.5708, -1, -0.1639, 0.9865)$ is the best known solution of the problem provided in [42].

Problem name: `Outrata1990Ex1a`

Source: [43]

Description: Outrata1990Ex1a is defined as follows

$$
\begin{aligned}
F(x,y) &:= \tfrac{1}{2}(y_1^2 + y_2^2) - 3y_1 - 4y_2 + r(x_1^2 + x_2^2) \\
f(x,y) &:= \tfrac{1}{2}\langle y, Hy\rangle - \langle b(x), y\rangle \\
g(x,y) &:= \begin{bmatrix} -0.333y_1 + y_2 - 2 \\ y_1 - 0.333y_2 - 2 \\ -y \end{bmatrix}
\end{aligned}
$$

with $r := 0.1$, $H := \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $b(x) := x$.

Comment: Outrata1990Ex1b, Outrata1990Ex1c, Outrata1990Ex1d, and Outrata1990Ex1e are obtained by respectively replacing r, H, and $b(x)$ in the lower-level objective function of Outrata1990a by

$$
\begin{aligned}
&r := 1, \quad H := \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}, \quad \text{and} \quad b(x) := x, \\
&r := 0, \quad H := \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}, \quad \text{and} \quad b(x) := x, \\
&r := 0.1, \quad H := \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}, \quad \text{and} \quad b(x) := x, \\
&r := 0.1, \quad H := \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}, \quad \text{and} \quad b(x) := \begin{bmatrix} -1 & 2 \\ 3 & -3 \end{bmatrix} x.
\end{aligned}
$$

According to [43], the best known solutions for problems Outrata1990Ex1a, Outrata1990Ex1b, Outrata1990Ex1c, Outrata1990Ex1d, and Outrata1990Ex1e are respectively

$$
\begin{aligned}
&(0.97, 3.14, 2.6, 1.8), \quad (0.28, 0.48, 2.34, 1.03), \\
&(20.26, 42.81, 3, 3), \quad (2, 0.06, 2, 0), \quad \text{and} \quad (2.42, -3.65, 0, 1.58).
\end{aligned}
$$

Note that for Outrata1990Ex1b and Outrata1990Ex1c, the solutions above change with a different starting point for the algorithm used in [43].

Problem name: `Outrata1990Ex2a`

Source: [43]

Description: Outrata1990Ex2a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}\left[(y_1-3)^2+(y_2-4)^2\right] \\ G(x,y) & := & -x \\ f(x,y) & := & \frac{1}{2}\langle y, H(x)y\rangle-(3+1.333x)y_1-xy_2 \\ g(x,y) & := & \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y \end{bmatrix} \end{array}$$

with the matrix $H(x)$ defined by $H(x) := I$.

Comment: Outrata1990Ex2b, Outrata1990Ex2c, Outrata1990Ex2d and Outrata1990Ex2e are respectively obtained by performing some changes on the terms $H(x)$ and $g(x,y)$ in the lower-level objective function of Outrata1990Ex2a:

$$H(x) := \begin{bmatrix} 1+x & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y \end{bmatrix},$$

$$H(x) := \begin{bmatrix} 1+x & 0 \\ 0 & 1+0.1x \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y \end{bmatrix},$$

$$H(x) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{bmatrix} (-0.333+0.1x)y_1+y_2-x \\ y_1+(-0.333-0.1x)y_2-2 \\ -y \end{bmatrix},$$

$$H(x) := \begin{bmatrix} 1+x & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{bmatrix} (-0.333+0.1x)y_1+y_2-x \\ y_1+(-0.333-0.1x)y_2-2 \\ -y \end{bmatrix},$$

According to [43], the best known solutions for problems Outrata1990Ex2a, Outrata1990Ex2b, Outrata1990Ex2c, Outrata1990Ex2d, Outrata1990Ex2e are respectively

$$\begin{array}{l} (2.07,3,3), \quad (0,3,3), \quad (3.456,1.707,2.569), \\ (2.498,3.632,2.8) \ \text{ and } \ (3.999,1.665,3.887), . \end{array}$$

Problem name: `Outrata1993Ex31`

Source: [44]

Description: Outrata1993Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2 \\ G(x,y) & := & -x \\ f(x,y) & := & \frac{1}{2}(1+0.2x)y_1^2+\frac{1}{2}(1+0.1x)y_2^2-(3+1.33x)y_1-xy_2 \\ g(x,y) & := & \begin{bmatrix} (-0.333+0.1x)y_1+y_2+0.1x-2 \\ y_1+(-0.333-0.1x)y_2+0.1x-2 \\ -y \end{bmatrix} \end{array}$$

Comment: `Outrata1993Ex32` is obtained by replacing the lower-level constraint by

$$g(x,y) \quad := \quad \begin{bmatrix} -0.333y_1+y_2+0.1x-1 \\ y_1^2+y_2^2-0.1x-9 \\ -y \end{bmatrix}.$$

For `Outrata1993Ex1` and `Outrata1993Ex2`, the best known solutions from [44] are $(1.90910, 2.97836, 2.23182)$ and $(4.06095, 2.68227, 1.48710)$, respectively.

Problem name: `Outrata1994Ex31`

Source: [45]
Description: Outrata1994Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2 \\ G(x,y) & := & \begin{bmatrix} -x \\ x-10 \end{bmatrix} \\ f(x,y) & := & \frac{1}{2}(1+0.2x)y_1^2+\frac{1}{2}(1+0.1x)y_2^2-(3+1.333x)y_1-xy_2 \\ g(x,y) & := & \begin{bmatrix} -0.333y_1+y_2+0.1x-1 \\ y_1^2+y_2^2-0.1x-9 \\ -y \end{bmatrix} \end{array}$$

Comment: According to [45], the best known solution is $(4.0604, 2.6822, 1.4871)$.

Problem name: `OutrataCervinka2009`
Source: [46]
Description: OutrataCervinka2009 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -2x_1-0.5x_2-y_2 \\ G(x,y) & := & x_1 \\ f(x,y) & := & y_1-y_2+x^\top y+\frac{1}{2}y^\top y \\ g(x,y) & := & \begin{bmatrix} y_2 \\ y_2-y_1 \\ y_2+y_1 \end{bmatrix} \end{array}$$

Comment: The point 0_4 is the global optimal solution of the problem according to [46].

Problem name: `PaulaviciusEtal2017a`
Source: [47]
Description: PaulaviciusEtal2017a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \\ -y-1 \\ y-1 \end{bmatrix} \\ f(x,y) & := & xy^2-\frac{1}{2}y^4 \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: This problem a slight modification of MitsosBarton2006Ex318, just with the upper-level objective function there replaced by x^2+y^2. Doing so, the point $(0.5, 0)$ remains globally optimal for the new problem [47].

Problem name: `PaulaviciusEtal2017b`
Source: [47]

Description: PaulaviciusEtal2017b is defined as follows

$$\begin{aligned} F(x,y) &:= x+y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \\ -y-1 \\ y-1 \end{bmatrix} \\ f(x,y) &:= 0.5xy^2 - x^3y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: This problem a slight modification of MitsosBarton2006Ex313, just with the *minus* in upper-level objective function replaced by a *plus*. Doing so, the global optimal solution the new problem above is $(-1,-1)$ according to [47].

Problem name: `SahinCiric1998Ex2`
Source: [48]
Description: SahinCiric1998Ex2 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-3)^2 + (y-2)^2 \\ G(x,y) &:= \begin{bmatrix} -x \\ x-8 \end{bmatrix} \\ f(x,y) &:= (y-5)^2 \\ g(x,y) &:= \begin{bmatrix} -2x+y-1 \\ x-2y+2 \\ x+2y-14 \end{bmatrix} \end{aligned}$$

Comment: The optimal value for the upper-level objective function is 5 and a corresponding global optimal point is $(1,3)$; cf. [48].

Problem name: `ShimizuAiyoshi1981Ex1`
Source: [49]
Description: ShimizuAiyoshi1981Ex1 is defined as follows

$$\begin{aligned} F(x,y) &:= x^2 + (y-10)^2 \\ G(x,y) &:= \begin{bmatrix} -15+x \\ y-x \\ -x \end{bmatrix} \\ f(x,y) &:= (x+2y-30)^2 \\ g(x,y) &:= \begin{bmatrix} x+y-20 \\ y-20 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(10,10)$ according to [49].

Problem name: `ShimizuAiyoshi1981Ex2`
Source: [49]

Description: ShimizuAiyoshi1981Ex2 is defined as follows

$$\begin{aligned} F(x,y) &:= (x_1-30)^2+(x_2-20)^2-20y_1+20y_2 \\ G(x,y) &:= \begin{bmatrix} -x_1-2x_2+30 \\ x_1+x_2-25 \\ x_2-15 \end{bmatrix} \\ f(x,y) &:= (x_1-y_1)^2+(x_2-y_2)^2 \\ g(x,y) &:= \begin{bmatrix} y-10_2 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(20,5,10,5)$ according to [49].

Problem name: `ShimizuEtal1997a`

Source: [50]

Description: ShimizuEtal1997a is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^2+(2y+1)^2 \\ f(x,y) &:= (y-1)^2-1.5xy \\ g(x,y) &:= \begin{bmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \end{bmatrix} \end{aligned}$$

Comment: Solutions are unknown. A possible solution is $(5,2)$.

Problem name: `ShimizuEtal1997b`

Source: [50]

Description: ShimizuEtal1997b is defined as follows

$$\begin{aligned} F(x,y) &:= 16x^2+9y^2 \\ G(x,y) &:= \begin{bmatrix} -4x+y \\ -x \end{bmatrix} \\ f(x,y) &:= (x+y-20)^4 \\ g(x,y) &:= \begin{bmatrix} 4x+y-50 \\ -y \end{bmatrix} \end{aligned}$$

Comment: $(11.25,5)$ is the global optimal solution of the problem and $(7.2,12.8)$ is a local optimal solution [50].

Problem name: `SinhaMaloDeb2014TP3`

Source: [51]

Description: SinhaMaloDeb2014TP3 is defined as follows

$$\begin{aligned} F(x,y) &:= -x_1^2-3x_2^2-4y_1+y_2^2 \\ G(x,y) &:= \begin{bmatrix} x_1^2+2x_2-4 \\ -x \end{bmatrix} \\ f(x,y) &:= 2x_1^2+y_1^2-5y_2 \\ g(x,y) &:= \begin{bmatrix} -x_1^2+2x_1-x_2^2+2y_1-y_2-3 \\ -x_2-3y_1+4y_2+4 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are -18.6787 and -1.0156, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP6`

Source: [51]

Description: SinhaMaloDeb2014TP6 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-1)^2 + 2y_1 - 2x \\ G(x,y) & := & -x \\ f(x,y) & := & (2y_1-4)^2 + (2y_2-1)^2 + xy_1 \\ g(x,y) & := & \begin{bmatrix} 4x+5y_1+4y_2-12 \\ 4y_2-4x-5y_1+4 \\ 4x-4y_1+5y_2-4 \\ 4y_1-4x+5y_2-4 \\ -y \end{bmatrix} \end{array}$$

Comment: The best known values of the upper-level and lower-level objective values are -1.2091 and 7.6145, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP7`

Source: [51]

Description: SinhaMaloDeb2014TP7 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -\frac{(x_1+y_1)(x_2+y_2)}{1+x_1y_1+x_2y_2} \\ G(x,y) & := & \begin{bmatrix} x_1^2+x_2^2-100 \\ x_1-x_2 \\ -x \end{bmatrix} \\ f(x,y) & := & \frac{(x_1+y_1)(x_2+y_2)}{1+x_1y_1+x_2y_2} \\ g(x,y) & := & \begin{bmatrix} y-x \\ -y \end{bmatrix} \end{array}$$

Comment: The best known values of the upper-level and lower-level objective values are -1.96 and 1.96, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP8`

Source: [51]

Description: SinhaMaloDeb2014TP8 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & |2x_1+2x_2-3y_1-3y_2-60| \\ g(x,y) & := & \begin{bmatrix} x_1+x_2+y_1-2y_2-40 \\ x-50_2 \\ -x \end{bmatrix} \\ f(x,y) & := & (y_1-x_1+20)^2 + (y_2-x_2+20)^2 \\ g(x,y) & := & \begin{bmatrix} 2y-x+10_2 \\ y-20_2 \\ -y-10_2 \end{bmatrix} \end{array}$$

Comment: The global optimal values of the upper-level and lower-level objective values are 0 and 100.0, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP9`

Source: [51]

Description: SinhaMaloDeb2014TP9 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \sum_{i=1}^{10}\left[(x_i-1)^2+y_i^2\right] \\ f(x,y) & := & \exp\left\{\left[1+\frac{1}{400}\sum_{i=1}^{10}y_i^2-\prod_{i=1}^{10}\cos\left(\frac{y_i}{\sqrt{i}}\right)\right]\sum_{i=1}^{10}x_i^2\right\} \\ g(x,y) & := & \begin{bmatrix} y-\pi_{10} \\ -y-\pi_{10} \end{bmatrix} \end{array}$$

Comment: The best known values of the upper-level and lower-level objective values are 0.0 and 1.0, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP10`

Source: [51]

Description: SinhaMaloDeb2014TP10 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \sum_{i=1}^{10}\left[(x_i-1)^2+y_i^2\right] \\ f(x,y) & := & \exp\left[1+\frac{1}{4000}\sum_{i=1}^{10}x_i^2y_i^2-\prod_{i=1}^{10}\cos\left(\frac{x_iy_i}{\sqrt{i}}\right)\right] \\ g(x,y) & := & \begin{bmatrix} y-\pi_{10} \\ -y-\pi_{10} \end{bmatrix} \end{array}$$

Comment: The best known values of the upper-level and lower-level objective values are 0.0 and 1.0, respectively; cf. [51].

Problem name: `TuyEtal2007`

Source: [52]

Description: TuyEtal2007 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & \begin{bmatrix} -x \\ -y \end{bmatrix} \\ f(x,y) & := & -y \\ g(x,y) & := & \begin{bmatrix} 3x+y-15 \\ x+y-7 \\ x+3y-15 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(4.492188, 1.523438)$; cf. [52].

Problem name: `Vogel2002`

Source: [53]

Description: Vogel2002 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (y+1)^2 \\ G(x,y) & := & \begin{bmatrix} -x-3 \\ x-2 \end{bmatrix} \\ f(x,y) & := & y^3-3y \\ g(x,y) & := & x-y \end{array}$$

Comment: The point $(-2,-2)$ is the global optimal solution of the problem; cf. [53].

Problem name: `WanWangLv2011`

Source: [54]

Description: WanWangLv2011 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (1+x_1-x_2+2y_2)(8-x_1-2y_1+y_2+5y_3) \\ f(x,y) & := & 2y_1-y_2+y_3 \\ g(x,y) & := & \left[\begin{array}{r} -y_1+y_2+y_3-1 \\ 2x_1-y_1+2y_2-0.5y_3-1 \\ 2x_2+2y_1-y_2-0.5y_3-1 \\ -x \\ -y \end{array}\right] \end{array}$$

Comment: The global optimal solution is $(0, 0.75, 0, 0.5, 0)$ according to [54].

Problem name: `YeZhu2010Ex42`

Source: [55]

Description: YeZhu2010Ex42 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-1)^2+y^2 \\ G(x,y) & := & \left[\begin{array}{r} -x-3 \\ x-2 \end{array}\right] \\ f(x,y) & := & y^3-3y \\ g(x,y) & := & x-y \end{array}$$

Comment: YeZhu2010Ex43 is obtained by replacing the upper-level objective function by

$$(x-1)^2+(y-2)^2$$

Note that YeZhu2010Ex42 is a slightly modified version of Vogel2002, with the term y^2 added to the upper-level objective function. The point $(1, 1)$ is the global optimal solution for both YeZhu2010Ex42 and YeZhu2010Ex43; cf. [55].

Problem name: `Yezza1996Ex31`

Source: [56]

Description: Yezza1996Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(4x-3)y+2x+1 \\ G(x,y) & := & \left[\begin{array}{r} -x \\ x-1 \end{array}\right] \\ f(x,y) & := & -(1-4x)y-2x-2 \\ g(x,y) & := & \left[\begin{array}{r} -y \\ y-1 \end{array}\right] \end{array}$$

Comment: The global optimal solution of the problem is $(0.25, 0)$; cf. [56].

Problem name: `Yezza1996Ex41`

Source: [56]

Description: Yezza1996Ex41 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y-2)^2+\frac{1}{2}(x-y-2)^2 \\ f(x,y) & := & \frac{1}{2}y^2+x-y \\ g(x,y) & := & \left[\begin{array}{r} -y \\ y-x \end{array}\right] \end{array}$$

Comment: The global optimal solution of the problem is $(3, 1)$; cf. [56].

Problem name: `Zlobec2001a`

Source: [57]

Description: Zlobec2001a is defined as follows

$$
\begin{aligned}
F(x,y) &:= -y_1/x \\
f(x,y) &:= -y_1 - y_2 \\
g(x,y) &:= \begin{bmatrix} x + y_1 \\ y_2 - 1 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: This example is used in [57] to illustrate that the objective function of the problem can be discontinuous. As stated in [57] , a global optimal solution is $(1, 1, 0)$.

Problem name: `Zlobec2001b`

Source: [57]

Description: Zlobec2001b is defined as follows

$$
\begin{aligned}
F(x,y) &:= x + y \\
G(x,y) &:= \begin{bmatrix} x - 1 \\ -x \end{bmatrix} \\
f(x,y) &:= -y \\
g(x,y) &:= \begin{bmatrix} y - 1 \\ -y \\ xy \\ -xy \end{bmatrix}
\end{aligned}
$$

Comment: This example is used in [57] to illustrate that the feasible set of a bilevel optimization problem is not necessarily closed. As stated in [57], this problem does not have an optimal solution.

Problem name: `DesignCentringP1`

Source: [71]

Description: DesignCentringP1 is a so-called design centring problem. The following model is built from Problem 1 in [71, Section 5.3] but with different G, f, g and y.

$$
\begin{aligned}
F(x,y) &:= -\pi x_3^2 \\
G(x,y) &:= \begin{bmatrix} -y_1 - y_2^2 \\ \frac{1}{4}y_3 + y_4 - \frac{3}{4} \\ -y_6 - 1 \end{bmatrix} \\
f(x,y) &:= y_1 + y_2^2 - \frac{1}{4}y_3 - y_4 + y_6 \\
g(x,y) &:= \begin{bmatrix} (y_1 - x_1)^2 + (y_2 - x_2)^2 - x_3^2 \\ (y_3 - x_1)^2 + (y_4 - x_2)^2 - x_3^2 \\ (y_5 - x_1)^2 + (y_6 - x_2)^2 - x_3^2 \end{bmatrix}
\end{aligned}
$$

Comment: We construct the lower level objectve function by $f(x,y) = -\sum_i G_i(x,y)$. The reported optimal value of the upper-level objective is $F(x,y) = 1.8606$; cf. [71]. However, since the whole problem is altered, the optimal value of the upper-level objective might be different. A possible solution is

$$(0.7486, -0.2304, 0.7696, -0.0084, -0.0917, 0.9352, 0.5162, 0.7486, -1)$$

with $F(x,y) = -1.4319$, $f(x,y) = -1.7500$.

Problem name: `DesignCentringP2`

Source: [71]

Description: DesignCentringP2 is built from Problem 2 in [71, Section 5.3] but with different G, f, g and y.

$$
\begin{aligned}
F(x,y) &:= -\pi x_3 x_4 \\
G(x,y) &:= \begin{bmatrix} -y_1 - y_2^2 \\ \frac{1}{4}y_3 + y_4 - \frac{3}{4} \\ -y_6 - 1 \\ x_3 - 1 \\ x_4 - 1 \end{bmatrix} \\
f(x,y) &:= y_1 + y_2^2 - \frac{1}{4}y_3 - y_4 + y_6 \\
g(x,y) &:= \begin{bmatrix} \frac{(y_1-x_1)^2}{x_3^2} + \frac{(y_2-x_2)^2}{x_4^2} - 1 \\ \frac{(y_3-x_1)^2}{x_3^2} + \frac{(y_4-x_2)^2}{x_4^2} - 1 \\ \frac{(y_5-x_1)^2}{x_3^2} + \frac{(y_6-x_2)^2}{x_4^2} - 1 \end{bmatrix}
\end{aligned}
$$

Comment: We construct the lower level objectve function by $f(x,y) = -\sum_i G_i(x,y)$. The reported optimal value of the upper-level objective is $F(x,y) = 3.4838$; cf. [71]. However, since the whole problem is altered, the optimal value of the upper-level objective might be different. A possible solution is $(3,0,1,1,3,0,3,0,3,0)$ with $F(x,y) = -\pi$, $f(x,y) = 2.25$.

Problem name: `DesignCentringP3`

Source: [71]

Description: DesignCentringP3 is built from Problem 3 in [71, Section 5.3] but with different G, f, g and y.

$$
\begin{aligned}
F(x,y) &:= -\pi \left| \det \begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \right| \\
G(x,y) &:= \begin{bmatrix} -y_1 - y_2^2 \\ \frac{1}{4}y_3 + y_4 - \frac{3}{4} \\ -y_6 - 1 \end{bmatrix} \\
f(x,y) &:= y_1 + y_2^2 - \frac{1}{4}y_3 - y_4 + y_6 \\
g(x,y) &:= \begin{bmatrix} \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \end{bmatrix}^\top \left(\begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} \right)^{-1} \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \end{bmatrix} - 1 \\ \begin{bmatrix} y_3 - x_1 \\ y_4 - x_2 \end{bmatrix}^\top \left(\begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} \right)^{-1} \begin{bmatrix} y_3 - x_1 \\ y_4 - x_2 \end{bmatrix} - 1 \\ \begin{bmatrix} y_5 - x_1 \\ y_6 - x_2 \end{bmatrix}^\top \left(\begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} \right)^{-1} \begin{bmatrix} y_5 - x_1 \\ y_6 - x_2 \end{bmatrix} - 1 \end{bmatrix}
\end{aligned}
$$

Comment: We construct the lower level objectve function by $f(x,y) = -\sum_i G_i(x,y)$. The reported optimal value of the upper-level objective is $F(x,y) = 3.7234$; cf. [71]. However, since the whole problem is altered, the optimal value of the upper-level objective might be different. This problem is a very challenging one whose first, second order derivatives of g are very complicated.

Problem name: `DesignCentringP4`

Source: [71]

Description: DesignCentringP4 is built from Problem 4 in [71, Section 5.3] but with different G, f, g and y.

$$
\begin{aligned}
F(x,y) &:= -(x_1-x_3)(x_2-x_4)\\
G(x,y) &:= \begin{bmatrix} -y_1-y_2^2 \\ \frac{1}{4}y_3+y_4-\frac{3}{4} \\ -y_6-1 \end{bmatrix}\\
f(x,y) &:= y_1+y_2^2-\tfrac{1}{4}y_3-y_4+y_6\\
g(x,y) &:= \begin{bmatrix} y-[x_1,x_2,x_1,x_2,x_1,x_2]^\top \\ -y+[x_3,x_4,x_3,x_4,x_3,x_4]^\top \end{bmatrix}
\end{aligned}
$$

Comment: We construct the lower level objectve function by $f(x,y) = -\sum_i G_i(x,y)$. The reported optimal value of the upper-level objective is $F(x,y) = 3.0792$; cf. [71]. However, since the whole problem is altered, the optimal value of the upper-level objective might be different. A possible solution is $(-1,1,-1,1,-1,1,-1,1,-1,1)$ with $F(x,y) = 0, f(x,y) = 0.25$.

Problem name: `OptimalControl`

Source: [37]

Description: `OptimalControl` is a bilevel optimal control program from [37]. This is a quadratic program and its dimensions $\{n_x, n_y, n_G, n_g\}$ are able to be altered. The model below is constructed based on the MATLAB M-file that the author P. Mehlitz in [37] shared with us.

$$
\begin{aligned}
F(x,y) &:= \frac{1}{2}\left(\begin{bmatrix} y^1 \\ 0 \end{bmatrix} - c\right)^\top M \left(\begin{bmatrix} y^1 \\ 0 \end{bmatrix} - c\right) - k^\top x,\\
G(x,y) &:= \begin{bmatrix} -x_1+x_2-1 \\ -x \end{bmatrix},\\
f(x,y) &:= \tfrac{1}{2}(Cy^1-Px)^\top W(Cy^1-Px) + \tfrac{\sigma}{2}(y^2-Qx)^\top U(y^2-Qx),\\
g(x,y) &:= \begin{bmatrix} y^2-u \\ -y^2+l \\ Ay \\ -Ay \end{bmatrix}.
\end{aligned}
$$

Comment: $y^\top = ((y^1)^\top\ (y^2)^\top)^\top$ with $y^i \in \mathbb{R}^{m_i}$ and $m_1+m_2 = n_y$, $M \in \mathbb{R}^{m\times m}, c \in \mathbb{R}^m, k \in \mathbb{R}^{n_x}, C \in \mathbb{R}^{s\times m_1}, P \in \mathbb{R}^{s\times n}, W \in \mathbb{R}^{s\times s}, Q \in \mathbb{R}^{m_2\times n_x}, U \in \mathbb{R}^{m_2\times m_2}, u \in \mathbb{R}^{m_2}, l \in \mathbb{R}^{m_2}, A \in \mathbb{R}^{t\times n_y}$ are given data. Particularly, in the MATLAB M-file, $n_x = 2, n_y = t = 2\texttt{ni}, n_G = 3, n_g = 4\texttt{ni}, m_1 = m_2 = s = \texttt{ni}$, where `ni` and $m >$ `ni` are two large numbers.

Problem name: `NetworkDesignP1`

Source: [11]

Description: NetworkDesignP1 is a network design problem from `NDP1` in [11, Section 4.5] defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & \left[50+\frac{y_1}{1+x_1}, 10\frac{y_2}{1+x_2}, 10+\frac{y_3}{1+x_3}, 10\frac{y_4}{1+x_4}, 50+\frac{x_5}{1+x_5}\right] y + 100\sum_{i=1}^{5} x_i \\
G(x,y) & := & -x-1 \\
f(x,y) & := & \int_0^{y_1}\left(50+\frac{u}{1+x_1}\right)du + \int_0^{y_2}\left(10\frac{u}{1+x_2}\right)du + \int_0^{y_3}\left(10+\frac{u}{1+x_3}\right)du \\
 & & +\int_0^{y_4}\left(10\frac{u}{1+x_4}\right)du + \int_0^{y_5}\left(50+\frac{u}{1+x_5}\right)du \\
g(x,y) & := & \begin{bmatrix} \phi(x,y) \\ -\phi(x,y) \\ -y \end{bmatrix} \text{ where } \phi(x,y) := \begin{bmatrix} -6+y_1+y_3+y_5 \\ y_2-y_5-y_3 \\ y_4-y_1-y_3 \end{bmatrix}
\end{array}
$$

Comment: The best known values of upper-level and lower-level objectives are $F(x,y) = 300.5$ and $f(x,y) = 419.8$; cf. [8].

Problem name: `NetworkDesignP2`

Source: [11]

Description: NetworkDesignP2 is a network design problem from `NDP2` in [11, Section 4.5] defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & \left[50+\frac{y_1}{1+x_1}, 10\frac{y_2}{1+x_2}, 10+\frac{y_3}{1+x_3}, 10\frac{y_4}{1+x_4}, 50+\frac{x_5}{1+x_5}\right] y + \sum_{i=1}^{5} x_i \\
G(x,y) & := & -x-1 \\
f(x,y) & := & \int_0^{y_1}\left(50+\frac{u}{1+x_1}\right)du + \int_0^{y_2}\left(10\frac{u}{1+x_2}\right)du + \int_0^{y_3}\left(10+\frac{u}{1+x_3}\right)du \\
 & & +\int_0^{y_4}\left(10\frac{u}{1+x_4}\right)du + \int_0^{y_5}\left(50+\frac{u}{1+x_5}\right)du \\
g(x,y) & := & \begin{bmatrix} \phi(x,y) \\ -\phi(x,y) \\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} -6+y_1+y_3+y_5 \\ y_2-y_5-y_3 \\ y_4-y_1-y_3 \end{bmatrix}
\end{array}
$$

Comment: For Network2 the best known values of upper-level and lower-level objectives are $F(x,y) = 142.9$ and $f(x,y) = 81.95$; cf. [8].

Problem name: `RobustPortfolioP1`

Source: [71]

Description: RobustPortfolioP1 is a portfolio problem which has been used in [71] to illustrate the robustness of an optimization problem. Based on Problem 6 in [71, Section 5.3], we build (P) in the following form

$$
\begin{array}{rcl}
F(x,y) & := & -x_{N+1} \\
G(x,y) & := & \begin{bmatrix} x_{N+1} - y^\top x \\ -x_{1:N} \\ \sum_{i=1}^{N} x_i - 1 \\ -\sum_{i=1}^{N} x_i + 1 \end{bmatrix} \\
f_j(x,y^j) & := & y^\top x_{1:N} - x_{N+1} \\
g(x,y) & := & \begin{bmatrix} \|\text{diag}\,(\sigma)^{-1}(y-\bar{y})\|_\delta^\delta - \theta^\delta \\ -y \end{bmatrix} \\
\bar{y}_i & := & 1.15 + \frac{0.05}{N} i \quad \text{for} \quad i = 1, ..., N \\
\sigma_i & := & \frac{0.05}{3N}\sqrt{2N(N+1)i} \quad \text{for} \quad i = 1, ..., N \\
\theta & := & 1.5 \\
\delta & \in & [1, \infty]
\end{array}
$$

Comment: Here, $\|y\|_\delta^\delta = (\sum_i |y_i|^\delta)^{1/\delta}$ for $\delta \in [1, +\infty]$. The upper-level variable is $x = (x_{1:N}; x_{N+1})$ with $x_{1:N} \in \mathbb{R}^N$. The scenarios considered for N are $N = 10, 50, 100, 150$. The equality constraint $H(x, y) := \sum_{i=1}^N x_i - 1$ is moved to $G(x, y)$.

Comment2: When $\delta = 2$, the optimal solution is $x_i = 1/N, y_i = 1.15, i = 1, \cdots, N, x_{N+1} = 1.15$. This is same as Problem 5 in [71].

Problem name: `RobustPortfolioP2`

Source: [71]

Description: RobustPortfolioP2 is built from Problem 7 in [71, Section 5.3]. It has the form as

$$
\begin{array}{rcl}
F(x,y) & := & -x_{N+1} \\
G(x,y) & := & \begin{bmatrix} x_{N+1} - y^\top x \\ -x_{1:N} \\ \sum_{i=1}^N x_i - 1 \\ -\sum_{i=1}^N x_i + 1 \end{bmatrix} \\
f(x,y) & := & y^\top x_{1:N} - x_{N+1} \\
g(x,y) & := & \begin{bmatrix} \sum_{i=1}^N \frac{(y_i - \bar{y}_i)^2}{\sigma_i^2} - \left\{\theta\left[1 + \sum_{i=1}^N \left(x_i - \frac{1}{N}\right)^2\right]\right\}^2 \\ -y \end{bmatrix} \\
\bar{y}_i & := & 1.15 + \frac{0.05}{N} i \quad \text{for} \quad i = 1, ..., N \\
\sigma_i & := & \frac{0.05}{3N}\sqrt{2N(N+1)i} \quad \text{for} \quad i = 1, ..., N \\
\theta & := & 1.5
\end{array}
$$

Comment: $N = 10, 50, 100, 150$. A possible solution is $x_i = 1/N, y_i = 1.15, i = 1, \cdots, N$ and $x_{N+1} = 1.15$.

Problem name: `TollSettingP1`

Source: [11]

Description: TollSettingP1 is a toll-setting problem which is able to be defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & -(x_1 y_3 + x_2 y_4 + x_3 y_8) \\
G(x,y) & := & -x \\
f(x,y) & := & [2, 6, 5 + x_1, x_2, 4, 2, 6, x_3]y \\
g(x,y) & := & \begin{bmatrix} \phi(x,y) \\ -\phi(x,y) \\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_1 + y_2 + y_3 - 1 \\ y_4 + y_5 - y_1 \\ y_6 + y_7 - y_2 - y_4 \\ y_8 - y_5 - y_6 \\ y_3 + y_7 + y_8 - 1 \end{bmatrix}
\end{array}
$$

Comment: The best known values of F and F are $F(x, y) = -7$ and $f(x, y) = 12$, cf. [8], and $(7, 4, 6, 0, 0, 1, 0, 0, 0, 0, 0)$ is a possible solution.

Problem name: `TollSettingP2`

Source: [11]

Description: TollSettingP2 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -(x_1(y_1+y_2)+x_2(y_3+y_4)+x_3(y_5+y_6))\\
G(x,y) &:= -x\\
f(x,y) &:= [2x_1,2x_1,2x_2,2x_2,2x_3,2x_3,5,7,14,7,2,4,29,20,12,8,5,2]y\\
g(x,y) &:= \begin{bmatrix} \phi(x,y)\\ -\phi(x,y)\\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_7+y_8+y_9-1\\ y_{10}+y_{11}+y_{12}-1\\ y_{13}+y_{14}+y_{15}-1\\ y_{16}+y_{17}+y_{18}-1\\ y_1+y_5+y_{13}-y_7\\ y_2+y_6+y_{16}-y_{10}\\ y_3+y_{14}-y_1-y_8\\ y_4+y_{17}-y_2-y_{11}\\ y_{15}-y_3-y_5-y_9\\ y_{18}-y_4-y_6-y_{12} \end{bmatrix}
\end{aligned}
$$

Comment: The best known values of upper-level and lower-level objectives are $F(x,y) = -9$ and $f(x,y) = 32$; cf. [8]. $(0.5,4,4.5,0,0,0,0,1,0,1,0,0,0,0,1,0,0,1,0,0,1)$ is a possible solution but with $F = -4.5$ and $f = 32$.

Problem name: `TollSettingP3`

Source: [11]

Description: TollSettingP3 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -(x_1(y_1+10y_2)+x_2(y_3+10y_4)+x_3(y_5+10y_6))\\
G(x,y) &:= -x\\
f(x,y) &:= [2x_1,20x_1,2x_2,20x_2,2x_3,20x_3,5,7,14,7,2,4,29,20,12,8,5,2]y\\
g(x,y) &:= \begin{bmatrix} \phi(x,y)\\ -\phi(x,y)\\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_7+y_8+y_9-1\\ y_{10}+y_{11}+y_{12}-1\\ y_{13}+y_{14}+y_{15}-1\\ y_{16}+y_{17}+y_{18}-1\\ y_1+y_5+y_{13}-y_7\\ 10y_2+10y_6+y_{16}-y_{10}\\ y_3+y_{14}-y_1-y_8\\ 10y_4+y_{17}-10y_2-y_{11}\\ y_{15}-y_3-y_5-y_9\\ y_{18}-10y_4-10y_6-y_{12} \end{bmatrix}
\end{aligned}
$$

Comment: The best known values of upper-level and lower-level objectives are $F(x,y) = -24$ and $f(x,y) = 81$; cf. [8]. $(5,3.5,8.5,0,0,1,0,0,0,0,1,0,0,0,1,0,0,1,0,0,1)$ is a possible solution but with $F = -3.5$ and $f = 32$.

Problem name: `TollSettingP4`

Source: [11]

Description: TollSettingP4 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -(y_2+y_3)x_1-y_3x_2\\
f(x,y) &:= [8,3+2x_1,3+2x_1+2x_2,6]y\\
g(x,y) &:= \begin{bmatrix} \phi(x,y)\\ -\phi(x,y)\\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_1+y_2-1\\ y_3+y_4-1 \end{bmatrix}
\end{aligned}
$$

Comment: The best known values of upper-level and lower-level objectives are $F(x,y) = -8$ and $f(x,y) = 14$; cf. [8]. Two possible optimal solutions are $(2.5, -1, 0, 1, 1, 0)$ and $(10/3, -4/3, 1, 0, 0, 1)$ but both with $F = -4$, $f = 14$.

Problem name: `TollSettingP5`

Source: [11]

Description: TollSettingP5 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(y_2+y_3)x \\ f(x,y) & := & [8, 3+2x, 4+2x, 6]y \\ g(x,y) & := & \begin{bmatrix} \phi(x,y) \\ -\phi(x,y) \\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_1+y_2-1 \\ y_3+y_4-1 \end{bmatrix} \end{array}$$

Comment: The best known values of upper-level and lower-level objectives are $F(x,y) = -11$ and $f(x,y) = 34$; cf. [8]. Two possible optimal solutions are $(2.5, 0, 1, 0, 1)$ with $F = -2.5$, $f = 14$ and $(1, 0, 1, 1, 0)$ with $F = -2$, $f = 11$.

2. LINEAR BILEVEL EXAMPLES

Problem name: `AnandalinghamWhite1990`

Source: [58]

Description: AnandalinghamWhite1990 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -x-3y \\ G(x,y) & := & -x \\ f(x,y) & := & -x+3y \\ g(x,y) & := & \begin{bmatrix} -x-2y+10 \\ x-2y-6 \\ 2x-y-21 \\ x+2y-38 \\ -x+2y-18 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(16, 11)$; cf. [58].

Problem name: `Bard1984a`

Source: [75]

Description: Bard1984a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x+y \\ G(x,y) & := & -x \\ f(x,y) & := & -5x-y \\ g(x,y) & := & \begin{bmatrix} -x-0.5y+2 \\ -0.25x+y-2 \\ x+0.5y-8 \\ x-2y-4 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(8/9, 20/9)$; cf. [75].

Problem name: `Bard1984b`

Source: [75]
Description: Bard1984b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -5x - y \\ G(x,y) & := & -x \\ f(x,y) & := & y \\ g(x,y) & := & \begin{bmatrix} -x - 0.5y + 2 \\ -0.25x + y - 2 \\ x + 0.5y - 8 \\ x - 2y - 4 \\ -y \end{bmatrix} \end{array}$$

Comment: The reported optimal solution is $(7.2, 1.6)$; cf. [75].

Problem name: `Bard1991Ex2`
Source: [5]
Description: Bard1991Ex2 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -x + 10y_1 - y_2 \\ G(x,y) & := & -x \\ f(x,y) & := & -y_1 - y_2 \\ g(x,y) & := & \begin{bmatrix} x + y_1 - 1 \\ x + y_2 - 1 \\ y_1 + y_2 - 1 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(0, 0, 1)$; cf. [5].

Problem name: `BardFalk1982Ex2`
Source: [59]
Description: BarkFalk1982Ex2 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -2x_1 + x_2 + 0.5y_1 \\ G(x,y) & := & -x \\ f(x,y) & := & x_1 + x_2 - 4y_1 + y_2 \\ g(x,y) & := & \begin{bmatrix} -2x_1 + y_1 - y_2 + 2.5 \\ x_1 - 3x_2 + y_2 - 2 \\ x_1 + x_2 - 2 \\ -y \end{bmatrix} \end{array}$$

Comment: Authors in [59] claimed an optimal solution is $(1, 0, 0.5, 1)$; while authors in [79] stated the global optimal should be $(2, 0, 1.5, 0)$. And the latter is the correct result.

Problem name: `BenAyedBlair1990a`
Source: [76]

Description: BenAyedBlair1990a is defined as follows

$$
\begin{aligned}
F(x,y) &:= -1.5x - 6y_1 - y_2 \\
G(x,y) &:= \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\
f(x,y) &:= -y_1 - 5y_2 \\
g(x,y) &:= \begin{bmatrix} x + 3y_1 + y_2 - 5 \\ 2x + y_1 + 3y_2 - 5 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(1, 0, 1)$; cf. [76].

Problem name: `BenAyedBlair1990b`

Source: [76]

Description: Ben-AyedBlair1990b is defined as follows

$$
\begin{aligned}
F(x,y) &:= -x - y \\
G(x,y) &:= -x \\
f(x,y) &:= y \\
g(x,y) &:= \begin{bmatrix} -4x - 3y + 19 \\ x + 2y - 11 \\ 3x + y - 13 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(1, 5)$; cf. [76].

Problem name: `BialasKarwan1984a`

Source: [77]

Description: BialasKarwan1984a is defined as follows

$$
\begin{aligned}
F(x,y) &:= -x - y_2 \\
G(x,y) &:= -x \\
f(x,y) &:= -y_2 \\
g(x,y) &:= \begin{bmatrix} x + y_1 + y_2 - 3 \\ -x - y_1 + y_2 + 1 \\ -x + y_1 + y_2 - 1 \\ x - y_1 + y_2 - 1 \\ y_2 - 0.5 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution should be $(1.5, 1, 0.5)$.

Problem name: `BialasKarwan1984b`

Source: [77]

Description: BialasKarwan1984b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -y \\ G(x,y) & := & -x \\ f(x,y) & := & y \\ g(x,y) & := & \begin{bmatrix} -x-2y+10 \\ x+2y-38 \\ -x+2y-18 \\ x-2y-6 \\ 2x-y-21 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(16, 11)$; cf. [77].

Problem name: `CandlerTownsley1982`

Source: [60]

Description: CandlerTownsley1982 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -8x_1-4x_2+4y_1-40y_2-4y_3 \\ G(x,y) & := & -x \\ f(x,y) & := & x_1+2x_2+y_1+y_2+2y_3 \\ g(x,y) & := & \begin{bmatrix} -y_1+y_2+y_3+1 \\ 2x_1-y_1+2y_2-0.5y_3-1 \\ 2x_2+2y_1-y_2-0.5y_3-1 \\ -y \end{bmatrix} \end{array}$$

Comment: The gloabl optimal solution for this problem is $(0, 0.9, 0, 0.6, 0.4)$ with values of upper-level and lower-level objectives $F(x,y) = -29.2$ and $f(x,y) = 3.2$; cf. [66].

Problem name: `ClarkWesterberg1988`

Source: [61]

Description: ClarkWesterberg1988 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x-4y \\ f(x,y) & := & y \\ g(x,y) & := & \begin{bmatrix} -2x+y \\ 2x+5y-108 \\ 2x-3y+4 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(19, 14)$; cf. [61].

Problem name: `ClarkWesterberg1990b`

Source: [10]

Description: ClarkWesterberg1990b is defined as follows

$$
\begin{aligned}
F(x,y) &:= -x - 3y_1 + 2y_2 \\
G(x,y) &:= \begin{bmatrix} -x \\ x-8 \end{bmatrix} \\
f(x,y) &:= -y_1 \\
g(x,y) &:= \begin{bmatrix} -y_1 \\ y_1 - 4 \\ -2x + y_1 + 4y_2 - 16 \\ 8x + 3y_1 - 2y_2 - 48 \\ -2x + y_1 - 3y_2 + 12 \end{bmatrix}
\end{aligned}
$$

Comment: The best known optimal value for the upper-level objective function is -13 and a corresponding optimal point is $(5,4,2)$; cf. [10].

Problem name: `GlackinEtal2009`

Source: [62]

Description: GlackinEtal2009 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -2x_1 + 4x_2 + 3y \\
G(x,y) &:= \begin{bmatrix} x_1 - x_2 + 1 \\ -x \end{bmatrix} \\
f(x,y) &:= -y \\
g(x,y) &:= \begin{bmatrix} x_1 + x_2 + y - 4 \\ 2x_1 + 2x_2 + y - 6 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(1,2,0)$; cf. [62].

Problem name: `HaurieSavardWhite1990`

Source: [78]

Description: HaurieSavardWhite1990 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x + 5y \\
f(x,y) &:= -y \\
g(x,y) &:= \begin{bmatrix} -3x + 2y - 6 \\ 3x + 4y - 48 \\ 2x - 5y - 9 \\ -x - y + 8 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(12,3)$; cf. [78].

Problem name: `HuHuangZhang2009`

Source: [63]

Description: HuHuangZhang2009 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -4x - y_1 - y_2 \\
G(x,y) &:= -x \\
f(x,y) &:= -x - 3y_1 \\
g(x,y) &:= \begin{bmatrix} x + y_1 + y_2 - \frac{25}{9} \\ x + y_2 - 2 \\ y_1 + y_2 - \frac{8}{9} \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $\left(\frac{17}{9}, \frac{8}{9}, 0\right)$; cf. [63]. Note that there is an error in [63] where $x + y_1 - 2 \leq 0$ should be $x + y_2 - 2 \leq 0$.

Problem name: `LanWenShihLee007`

Source: [64]

Description: LanWenShihLee2007 is defined as follows

$$\begin{aligned} F(x,y) &:= 2x - 11y \\ G(x,y) &:= -x \\ f(x,y) &:= x + 3y \\ g(x,y) &:= \begin{bmatrix} x - 2y - 4 \\ 2x - y - 24 \\ 3x + 4y - 96 \\ x + 7y - 126 \\ -4x + 5y - 65 \\ -x - 4y + 8 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known solution is $(17.4500, 10.9080)$; cf. [64].

Problem name: `LiuHart1994`

Source: [67]

Description: LiuHart1994 is defined as follows

$$\begin{aligned} F(x,y) &:= -x - 3y \\ G(x,y) &:= -x \\ f(x,y) &:= y \\ g(x,y) &:= \begin{bmatrix} -x + y - 3 \\ x + 2y - 12 \\ 4x - y - 12 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The reported optimal solution is $(4, 4)$; cf. [67].

Problem name: `MershaDempe2006Ex1`

Source: [65]

Description: MershaDempe2006Ex1 is defined as follows

$$\begin{aligned} F(x,y) &:= x - 8y \\ G(x,y) &:= -x \\ f(x,y) &:= y \\ g(x,y) &:= \begin{bmatrix} 5x - 2y - 33 \\ -x - 2y + 9 \\ -7x + 3y - 5 \\ x + y - 15 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The original problem has no optimal solution, so the version used here is shifted the upper level constraints to the lower level; cf. [65].

Problem name: `MershaDempe2006Ex2`

Source: [65]

Description: MershaDempe2006Ex2 is defined as follows

$$\begin{aligned} F(x,y) &:= -x - 2y \\ G(x,y) &:= \begin{bmatrix} -2x + 3y - 12 \\ x + y - 14 \end{bmatrix} \\ f(x,y) &:= -y \\ g(x,y) &:= \begin{bmatrix} -3x + y + 3 \\ 3x + y - 30 \end{bmatrix} \end{aligned}$$

Comment: Reported global optimal solution is $(8, 6)$; cf. [65].

Problem name: `TuyEtal1993`

Source: [79]

Description: TuyEtal1993 is defined as follows

$$\begin{aligned} F(x,y) &:= -2x_1 + x_2 + 0.5y_1 \\ G(x,y) &:= \begin{bmatrix} x_1 + x_2 - 2 \\ -x \end{bmatrix} \\ f(x,y) &:= -4y_1 + y_2 \\ g(x,y) &:= \begin{bmatrix} -2x_1 + y_1 - y_2 + 2.5 \\ x_1 - 3x_2 + y_2 - 2 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The reported optimal solution is $(2, 0, 1.5, 0)$; cf. [79].

Problem name: `TuyEtal1994`

Source: [80]

Description: TuyEtal1994 is defined as follows

$$\begin{aligned} F(x,y) &:= 3x_1 + 2x_2 + y_1 + y_2 \\ G(x,y) &:= \begin{bmatrix} x_1 + x_2 + y_1 + y_2 - 4 \\ -x \end{bmatrix} \\ f(x,y) &:= 4y_1 + y_2 \\ g(x,y) &:= \begin{bmatrix} -3x_1 - 5x_2 - 6y_1 - 2y_2 + 15 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The reported global optimal solution is $(0, 3, 0, 0)$; cf. [80].

Problem name: `TuyEtal2007Ex3`

Source: [52]

Description: TuyEtal2007Ex3 is defined as follows

$$\begin{aligned} F(x,y) &:= (12, -1, -12, 13, 0, 2, 0, -5, 6, -11)x - (5, 6, 4, 7, 0, 0)y \\ G(x,y) &:= \begin{bmatrix} -Ax - By + [-30, 134]^\top \\ -x \end{bmatrix} \\ f(x,y) &:= [3, -2, -3, -3, 1, 6]y \\ g(x,y) &:= \begin{bmatrix} -Cx - Dy + [-83, -92, -168, 96, 133, -89, 192]^\top \\ -y \end{bmatrix} \end{aligned}$$

with A, B, C, and D respectively defined by

$$A := \begin{bmatrix} 2 & 3 & -14 & 2 & 9 & -2 & -1 & 4 & 0 & -2 \\ -1 & 7 & -13 & 0 & 15 & -2 & 8 & 4 & -4 & 7 \end{bmatrix}$$
$$B := \begin{bmatrix} 3 & -9 & 2 & 8 & -1 & 8 \\ 6 & 2 & -6 & -2 & -8 & 4 \end{bmatrix}$$
$$C := \begin{bmatrix} 5 & -7 & -4 & 2 & -3 & 9 & -9 & 1 & 3 & -11 \\ -6 & 5 & 3 & 2 & -8 & -5 & -8 & 3 & -7 & -3 \\ 6 & 4 & -2 & 0 & 2 & -3 & 3 & -2 & -2 & -4 \\ -5 & -6 & 0 & 4 & -3 & 8 & -1 & 0 & -2 & 3 \\ -11 & 11 & -4 & -5 & 10 & 6 & -14 & 7 & 11 & 3 \\ -9 & 12 & 4 & 10 & -2 & -8 & -5 & 11 & 4 & -1 \\ -7 & 2 & 6 & 0 & 11 & -1 & 2 & 2 & 1 & 2 \end{bmatrix}$$
$$D := \begin{bmatrix} -10 & 9 & 6 & -4 & -6 & 3 \\ 5 & 7 & -1 & -1 & 6 & -4 \\ -10 & -5 & -6 & 4 & -3 & 1 \\ 4 & 3 & 4 & 4 & -1 & -1 \\ 10 & 7 & -7 & -7 & -2 & -7 \\ -2 & 5 & -10 & -1 & -4 & -5 \\ 5 & 5 & 6 & 5 & -1 & 12 \end{bmatrix}$$

Comment: According to [52] the best known solution of the problem is

$$\begin{aligned} x^* &= (0, 8.170692, 10, 0, 7.278940, 3.042311, 0, 10, 0.001982, 9.989153) \\ y^* &= (3.101280, 10, 10, 10, 0, 9.846133) \end{aligned}$$

with $F = -467.4613$, $f = -11.6194$.

Problem name: `VisweswaranEtal1996`

Source: [68]

Description: VisweswaranEtal1996 is defined as follows

$$\begin{aligned} F(x,y) &:= x + y \\ G(x,y) &:= -x \\ f(x,y) &:= -5x - y \\ g(x,y) &:= \begin{bmatrix} -x - 0.5y + 2 \\ -0.25x + y - 2 \\ x + 0.5y - 8 \\ x - 2y - 2 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The reported optimal solution is $(8/9, 20/9)$; cf. [68].

Problem name: `WangJiaoLi2005`

Source: [66]

Description: WangJiaoLi2005 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -100x - 1000y_1\\
G(x,y) &:= \begin{bmatrix} -x \\ x-1 \end{bmatrix}\\
f(x,y) &:= -y_1 - y_2\\
g(x,y) &:= \begin{bmatrix} x+y_1-y_2-1 \\ y_1+y_2-1 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: The reported optimal values of the objective functions are $F(x,y) = -1000$ and $f(x,y) = -1$; cf. [66].

3. Simple bilevel examples

Problem name: `FrankeEtal2018Ex53`

Source: [73]

Description: FrankeEtal2018Ex53 examples are defined as follows

$$
\begin{aligned}
F(y) &:= y_1^2 + y_2^2\\
G(y) &:= \begin{bmatrix} -y \\ y-1_2 \end{bmatrix}\\
f(y) &:= (y_1-2)^2\\
g(y) &:= \begin{bmatrix} -y \\ y-1_2 \end{bmatrix}
\end{aligned}
$$

Comment: The reported global optimal solution is $(1, 0)$; cf. [73].

Problem name: `FrankeEtal2018Ex511`

Source: [73]

Description: FrankeEtal2018Ex511 examples are defined as follows

$$
\begin{aligned}
F(y) &:= 0.5(y_1-2)^2 + 0.5y_2^2 + 0.5(y_3-2)^2\\
f(y) &:= y_1 + y_2 + y_3\\
g(y) &:= \begin{bmatrix} -y_1-y_2 \\ -y_1+y_2 \\ -y_1 \\ -y_3 \end{bmatrix}
\end{aligned}
$$

Comment: The reported global optimal solution is $(1, -1, 0)$; cf. [73].

Problem name: `FrankeEtal2018Ex513`

Source: [73]

Description: FrankeEtal2018Ex513 examples are defined as follows

$$
\begin{aligned}
F(y) &:= -y_2\\
f(y) &:= y_3\\
g(y) &:= \begin{bmatrix} y_1^2 - y_3 \\ y_1^2 + y_2^2 - 1 \\ -y_3 \end{bmatrix}
\end{aligned}
$$

Comment: The reported global optimal solution is $(0, 1, 0)$; cf. [73].

Problem name: `FrankeEtal2018Ex521`

Source: [73]

Description: FrankeEtal2018Ex521 examples are defined as follows

$$\begin{aligned} F(y) &:= -y_2 \\ f(y) &:= y_1 \\ g(y) &:= \begin{bmatrix} (y_1-1)^2-(y_2-0.5)^2-1.25 \\ y_1+y_2-1 \\ -y_1 \end{bmatrix} \end{aligned}$$

Comment: The reported global optimal solution is $(0,1)$; cf. [73].

Problem name: `MitsosBarton2006Ex31`

Source: [39]

Description: MitsosBarton2006Ex31 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \\ f(y) &:= -y \\ g(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The optimal solution is 1; cf. [39].

Problem name: `MitsosBarton2006Ex32`

Source: [39]

Description: MitsosBarton2006Ex32 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-1 \\ y-1 \\ y \end{bmatrix} \\ f(y) &:= -y \\ g(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The problem has no optimal solution; cf. [39].

Problem name: `MitsosBarton2006Ex33`

Source: [39]

Description: MitsosBarton2006Ex33 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-10 \\ y-10 \end{bmatrix} \\ f(y) &:= y^2 \\ g(y) &:= \begin{bmatrix} 1-y^2 \\ -y-10 \\ y-10 \end{bmatrix} \end{aligned}$$

Comment: The optimal solution is -1; cf. [39].

Problem name: `MitsosBarton2006Ex34`

Source: [39]

Description: MitsosBarton2006Ex34 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-0.5 \\ y-1 \end{bmatrix} \\ f(y) &:= -y^2 \\ g(y) &:= \begin{bmatrix} -y-0.5 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The optimal solution is 1; cf. [39].

Problem name: `MitsosBarton2006Ex35`

Source: [39]

Description: MitsosBarton2006Ex35 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \\ f(y) &:= 16y^4+2y^3-8y^2-1.5y+0.5 \\ g(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The optimal solution is 0.5; cf. [39].

Problem name: `MitsosBarton2006Ex36`

Source: [39]

Description: MitsosBarton2006Ex36 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \\ f(y) &:= y^3 \\ g(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

The optimal solution is -1; cf. [39].

Problem name: `ShehuEtal2019Ex42`

Source: [72]

Description: ShehuEtal2019Ex42 examples are defined as follows

$$\begin{aligned} F(y) &:= 0.5y^TQy \\ f(y) &:= 0.5\|Ay-b\|_2^2+\mu\|y\|_1 \end{aligned}$$

Comment1: Q is a positive definite matrix. $A \in \mathbb{R}^{m\times n_y}$ is a given matrix, b is a given vector and μ (e.g., 0.5) is positive scalar. b is generated as $b = Ay + \epsilon\zeta$, where A and ζ are random matrices whose elements are normally distributed with zero mean and variance 1, ϵ (e.g., 0.01) is a noisy factor. y is a generated sparse vector with few non-zero elements.

Acknowledgements We thank Dr P. Mehlitz for sharing with us their MATLAB M-file of example `IOC2019`.

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