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A simple approach to optimality conditions in minmax programming

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Considering the minmax programming problem, lower and upper subdifferential optimality conditions, in the sense of Mordukhovich, are derived. The approach here, mainly based on the nonsmooth dual objects of Mordukhovich, is completely different from that of most of the previous works where generalizations of the alternative theorem of Farkas have been applied. The results obtained are close to those known in the literature. However, one of the main achievements of this article is that we could also derive necessary optimality conditions for the minmax program of the usual Karush–Kuhn–Tucker type, which seems to be new in this field of study.

Keywords: minmax programming; optimal value function; optimality conditions; inner semicompact/semicontinuous set-valued mapping; convexity; basic subdifferential and normal cone

AMS Subject Classifications: 49K35; 90C47; 90C30

1. Introduction

In this article, we consider the so-called minmax optimization problem

$$\text{(P)} \qquad \text{minimize } \varphi(x) \text{ subject to } x \in X,$$

where the objective function φ and feasible set X are given, respectively, as

$$\varphi(x) := \max_y \{f(x, y) |\, g(x, y) \leq 0\} \quad \text{and} \quad X := \{x \in \mathbb{R}^n |\, G(x) \leq 0\} \tag{1.1}$$

with the functions f: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, g: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ and G: $\mathbb{R}^n \to \mathbb{R}^k$. Problem (P) may also be called the outer/upper level/leader's problem whereas the parametric problem $\max_y\{f(x, y)|\, g(x, y) \leq 0\}$ is the inner problem. If the inequality $g(x, y) \leq 0$ is replaced by an inclusion $y \in \Psi(x)$, where $\Psi(x)$ stands for the solution set of a minimization problem parametrized in x, one obtains the pessimistic reformulation of the bilevel optimization problem [2]. The optimistic reformulation is obtained similarly but with a minimization in the inner problem. The latter problems are much

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more complicated. A detailed discussion of optimality conditions in optimistic and pessimistic bilevel programming, in the perspective of optimization problems with value function objectives, can be found in [4] and [5], respectively. For a more general overview of bilevel programming, the interested reader is referred to the monograph by Dempe [2].

The minmax problem is said to be static or nonparametric if the inner constraint $g(x, y) \leq 0$ is not perturbed, meaning that the outer problem is of the form

$$\min_{x \in X} \max_{y \in Y} f(x, y), \tag{1.2}$$

where Y is a nonempty closed set independent of x. This is the class of minmax problem most often investigated in the literature, see e.g. [15] and references therein or more recently, see the paper by Dhara and Mehra [7]. Ishizuka [8] was the first to investigate necessary optimality conditions for the parametric minmax program (P). Further results were obtained in his book with Shimizu and Bard [16].

Schmitendorf [15] and Ishizuka [8] and many other authors have used generalizations of Farkas' alternative theorem to derive necessary optimality conditions for the minmax problem. Considering the static min/max program (1.2), Dhara and Mehra first transformed it into a semi-infinite programming problem

$$\begin{aligned} \min z \quad & \text{s.t. } x \in X \\ & f(x, y) \leq z, \quad \forall y \in Y, \end{aligned}$$

which was then converted to an optimization problem with finitely many constraints. Approximate and limiting/lower subdifferential optimality were then derived for the latter problem. Already, the transformation process in [7] seems to us to be quite complicated.

In this article, we consider the more general minmax program (P). The intention is to write lower and upper subdifferential optimality conditions. To proceed, we take problem (P) as it is, that is, an optimization problem with value function objective and by a well-known result by Mordukhovich [10], a necessary optimality condition for a point $\bar{x}$ to solve (P) is that

$$0 \in \partial\varphi(\bar{x}) + N_X(\bar{x}) \tag{1.3}$$

provided φ is Lipschitz continuous near $\bar{x}$. Here, $\partial\varphi$ and N_X denote the basic/limiting/Mordukhovich subdifferential and normal cone, respectively. The definitions of these objects are given in the following section. The condition in (1.3) represents what Mordukhovich called lower subdifferential optimality conditions. This, in fact, induces the usual optimality conditions. Hence, from time to time, the prefix 'lower subdifferential' will be omitted in this article when such conditions are investigated. The only thing that remains to be done here is estimating the basic subdifferential of φ and the basic normal cone to X.

With the structure of (P), it may well happen that the objective function φ be concave. Hence, according to Mordukhovich [9], upper subdifferential optimality conditions may be more suitable in such a case. An upper subdifferential necessary condition for $\overline{x}$ to be a local optimal solution of (P) is that

$$-\widehat{\partial}^{+}\varphi(\overline{x}) \subseteq \widehat{N}_X(\overline{x}) \tag{1.4}$$

provided φ is finite at $\overline{x}$ [9]. In (1.4), $\widehat{\partial}^{+}\varphi$ and $\widehat{N}_X$ denote the Fréchet subdifferential and normal cone, respectively. Their definitions are also provided in the following section.

We have applied the approach in (1.3) for various instances of the problem data. Namely, we have derived (lower subdifferential) optimality conditions for the minmax problem (P) when the functions f, g and G are all locally Lipschitz continuous (see Section 3), all smooth (see Section 5) and $-f$, g_i, $i=1,\dots,p$ are all convex (see Section 4). In the smooth case, upper subdifferential optimality conditions were derived as well. In the smooth static case, the lower subdifferential optimality conditions obtained here are close to those of [15] whereas in the smooth parametric case, the upper conditions obtained have the same structure as those of [8]. They even coincide in some cases. More interestingly, we have got Karush–Kuhn–Tucker (KKT)-type optimality conditions for the minmax problem, in the usual sense, that is, with no number attached to the gradient/generalized gradient of the objective function f, cf. Theorem 3.2, Theorem 4.1 and Corollary 5.2. To the best of our knowledge, such conditions have not been obtained before.

The constraint qualifications (CQs) used in this article are dual versions of the Mangasarian–Fromovitz CQ (MFCQ) or its nonsmooth extensions. The Slater CQ is also applied when the functions are convex. Most of the notations and vocabulary are borrowed from Dempe et al. [3] and Mordukhovich [9]. Necessary basic tools from variational analysis are presented in the following section and this article is concluded in Section 6.

2. Tools from variational analysis

More details on the material briefly discussed in this section can be found in the books by Mordukhovich [10] and Rockafellar and Wets [14]. We start with the *Kuratowski–Painlevée outer/upper limit* of a set-valued mapping $\Psi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, which is defined at a point $\overline{x}$ as

$$\limsup_{x\to\overline{x}} \Psi(x) := \left\{v\in\mathbb{R}^m | \exists x_k \to \overline{x},\ v_k \to v \text{ with } v_k\in\Psi(x_k) \text{ as } k\to\infty\right\}. \tag{2.1}$$

For an extended real-valued function $\psi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, the *Fréchet/viscosity* (*lower*) *subdifferential* of ψ at a point $\overline{x}$ of its domain is given by

$$\widehat{\partial}\psi(\overline{x}) := \left\{v\in\mathbb{R}^n | \liminf_{x\to\overline{x}} \frac{\psi(x)-\psi(\overline{x})-\langle v, x-\overline{x}\rangle}{\|x-\overline{x}\|} \geq 0\right\},$$

whereas the the Fréchet/viscosity upper subdifferential of ψ at $\overline{x}$ is obtained as

$$\widehat{\partial}^{+}\psi(\overline{x}) := -\widehat{\partial}(-\psi)(\overline{x}). \tag{2.2}$$

As for the *basic/limiting/Mordukhovich* (*lower*) *subdifferential* of ψ, it is the Kuratowski–Painlevée upper limit of the set-valued mapping $\widehat{\partial}\psi$ at $\overline{x}$:

$$\partial\psi(\overline{x}) := \limsup_{x\to\overline{x}} \widehat{\partial}\psi(x).$$

The *upper* counterpart of the basic subdifferential can be defined analogously to (2.2) via the lower basic subdifferential. If ψ is convex, then $\partial\psi(\overline{x})$ reduces to the subdifferential in the sense of convex analysis, that is

$$\partial\psi(\overline{x}) := \{v\in\mathbb{R}^n |\, \psi(x) - \psi(\overline{x}) \geq \langle v, x-\overline{x}\rangle,\ \forall x\in\mathbb{R}^n\}. \tag{2.3}$$

For a local Lipschitz continuous function, $\partial\psi(\overline{x})$ is nonempty and compact. Moreover, its convex hull is the subdifferential of Clarke, that is, one can define the Clarke subdifferential $\overline{\partial}\psi(\overline{x})$ of ψ at $\overline{x}$ by

$$\overline{\partial}\psi(\overline{x}) := \text{co}\,\partial\psi(\overline{x}), \tag{2.4}$$

where 'co' stands for the convex hull of the set in question. Thanks to this link between the Mordukhovich and Clarke subdifferentials, we have the following *convex hull property* which plays and important role in this article:

$$\text{co}\,\partial(-\psi)(\overline{x}) = -\text{co}\,\partial\psi(\overline{x}). \tag{2.5}$$

For this equality to hold, ψ should be Lipschitz continuous near $\overline{x}$.

The function ψ is said to be lower/subdifferentially/Clarke (resp. upper/supperdifferentially) regular at $\overline{x}$ if one has

$$\widehat{\partial}\psi(\overline{x}) = \partial\psi(\overline{x}) \quad (\text{resp. } \widehat{\partial}^{+}\psi(\overline{x}) = \partial^{+}\psi(\overline{x})). \tag{2.6}$$

Obviously, ψ is upper regular if and only if $-\psi$ is lower regular at the point in question.

We now introduce the *basic/limiting/Mordukhovich normal cone* to a set $\Omega\subseteq\mathbb{R}^n$ at one of its points $\overline{x}$

$$N_\Omega(\overline{x}) := \limsup_{x\to\overline{x}\,(x\in\Omega)} \widehat{N}_\Omega(\overline{x}) \tag{2.7}$$

where $\widehat{N}_\Omega(\overline{x})$ denotes the *prenormal/Fréchet normal cone* to Ω at $\overline{x}$ defined by

$$\widehat{N}_\Omega(\overline{x}) := \left\{v\in\mathbb{R}^n |\, \limsup_{x\to\overline{x}(x\in\Omega)} \frac{\langle v, x-\overline{x}\rangle}{\|x-\overline{x}\|} \leq 0\right\}$$

and 'limsup' stands for the Kuratowski–Painlevée upper limit defined in (2.1).

A set-valued mapping $\Psi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ will be said to be inner semicompact at a point $\overline{x}$, if for every sequence $x_k\to\overline{x}$, there is a sequence of $y_k\in\Psi(x_k)$ that contains a convergent subsequence as $k\to\infty$. It follows that the inner semicompactness holds whenever Ψ is uniformly bounded and has nonempty values around $\overline{x}$, i.e. there exists a neighbourhood U of $\overline{x}$ and a bounded set $\Omega\subset\mathbb{R}^m$ such that

$$\emptyset \neq \Psi(x) \subseteq \Omega, \quad \text{for all } x\in U. \tag{2.8}$$

The mapping Ψ is inner semicontinuous at $(\overline{x},\overline{y})\in\text{gph}\,\Psi$ if for every sequence $x_k\to\overline{x}$ there is a sequence of $y_k\in\Psi(x_k)$ that converges to $\overline{y}$ as $k\to\infty$. Obviously, if Ψ is closed-graph and inner semicompact at $\overline{x}$ with $\Psi(\overline{x}) = \{\overline{y}\}$, then Ψ is inner semicontinuous at $(\bar{x}, \bar{y})$. In general though, the inner semicontinuity is a property much stronger than the inner semicompactness and it is a necessary condition for the Lipschitz-like property to hold. If Ψ has a close graph, Ψ is Lipschitz-like around $(\bar{x}, \bar{y})$ if and only if the following coderivative/Mordukhovich criterion holds [10]:

$$D^*\Psi(\bar{x}, \bar{y})(0) = \{0\}. \tag{2.9}$$

For $(\bar{x}, \bar{y}) \in \text{gph}\,\Psi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m |\, y \in \Psi(x)\}$, the coderivative of Ψ at $(\bar{x}, \bar{y})$ is a homogeneous mapping $D^*\Psi(\bar{x}, \bar{y})$: $\mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined at $v \in \mathbb{R}^m$ by

$$D^*\Psi(\bar{x}, \bar{y})(v) := \{u \in \mathbb{R}^n | (u, -v) \in N_{\text{gph}\,\Psi}(\bar{x}, \bar{y})\}. \tag{2.10}$$

Here, $N_{\text{gph}\,\Psi}$ denotes the basic normal cone (2.7) to gphΨ. Finally, let us mention the calmness property that will also be useful in this article. The set-valued mapping Ψ will be said to be calm at some point $(\bar{x}, \bar{y}) \in \text{gph}\,\Psi$, if there exist neighbourhoods U of $\bar{x}$, V of $\bar{y}$, and a constant $\kappa > 0$ such that

$$\Psi(x) \cap V \subseteq \Psi(\bar{x}) + \kappa\|x - \bar{x}\|\mathbb{B}, \quad \forall x \in U$$

with $\mathbb{B}$ denoting the unit ball in $\mathbb{R}^m$. Ψ is automatically calm at $(\bar{x}, \bar{y})$, if it is Lipschitz-like around the same point.

3. Minmax programs with Lipschitzian initial data

In this section, we are mainly concerned by the minmax problem (P) in the case where all the functions f, g and G are locally Lipschitz continuous. Hence, by means of the nonsmooth objects of Mordukhovich and Clarke, described in the previous section, we will derive (lower subdifferential) necessary optimality conditions. The regularity rules needed here are well-known nonsmooth counterparts of the MFCQ, introduced by Mordukhovich [10] and Clarke [1], respectively. For the *upper level constraints* represented in problem (P) by the set X, we define the following *upper level regularity* at $\bar{x}$:

$$\left[0 \in \sum_{j=1}^{k} \alpha_j \partial G_j(\bar{x}), \quad \alpha_j \geq 0,\ \alpha_j G_j(\bar{x}) = 0,\ j = 1, \ldots, k\right] \Longrightarrow \alpha_j = 0,\ j = 1, \ldots, k. \tag{3.1}$$

A clear distinction has to be made between the 'upper level regularity', as a CQ and the 'upper regularity' of a function given in (2.6). For the *inner* or *lower level constraints* $g(x, y) \leq 0$, we first introduce the following independent CQs, that is

$$\left[0 \in \sum_{i=1}^{p} \beta_i \partial g_i(\bar{x}, \bar{y}),\ \beta_i \geq 0, \beta_i g_i(\bar{x}, \bar{y}) = 0,\ i = 1, \ldots, p\right] \Longrightarrow \beta_i = 0,\ i = 1, \ldots, p \tag{3.2}$$

and the next, which differs from the previous one by the fact that the multipliers must not vanish but the x-component of the sum $\sum_{i=1}^{p} \beta_i \partial g_i(\bar{x}, \bar{y})$:

$$\left[(x^*, 0) \in \sum_{i=1}^{p} \beta_i \partial g_i(\bar{x}, \bar{y}),\ \beta_i \geq 0,\ \beta_i g_i(\bar{x}, \bar{y}) = 0,\ i = 1, \ldots, p\right] \Longrightarrow x^* = 0. \tag{3.3}$$

Condition (3.3) combined with (3.2) implies the satisfaction of the coderivative criterion (2.9) for the set-valued mapping $K(x) := \{y \in \mathbb{R}^m | g(x, y) \leq 0\}$.

Hence, the fulfilment of the Lipschitz-like property for the latter multifunction, see [10] for more details on this issue and other related properties. We cannot close this list of basic-type CQs without mentioning the following stronger condition which automatically ensures that (3.2) and (3.3) hold:

$$\left[(u,0)\in\sum_{i=1}^{p}\beta_i\partial g_i(\overline{x},\overline{y}),\ \beta_i\geq 0,\ \beta_i g_i(\overline{x},\overline{y})=0,\ i=1,\ldots,p\right]\Longrightarrow \beta_i=0,\ i=1,\ldots,p. \tag{3.4}$$

Also of interest in this section and the subsequent ones are the inner semicompactness and inner semicontinuity of the *argminimum/solution* set-valued mapping

$$S(x):=\arg\min_y\{-f(x,y)|\, g(x,y)\leq 0\} \tag{3.5}$$

of the optimization problem $\min_y\{-f(x,y)|\, g(x,y)\leq 0\}$ parametrized in x. Obviously, if one sets the associated optimal value function as

$$\varphi_o(x):=\min_y\{-f(x,y)|\, g(x,y)\leq 0\} \tag{3.6}$$

then the value function φ(1.1) coincides with the negative of φ_o, that is

$$\varphi(x)=-\varphi_o(x),\quad \text{for all } x\in X. \tag{3.7}$$

We start with the necessary optimality conditions of the minmax program in the case where S (3.5) is inner semicompact.

Theorem 3.1 (Lower subdifferential optimality conditions in the nonsmooth case under the inner semicompactness of S) *Let $\overline{x}$ be an upper level regular local optimal solution for* (*P*), *where the functions f, g and G are all locally Lipschitz continuous. Furthermore, assume that S is inner semicompact around $\overline{x}$ and let CQs* (3.2) *and* (3.3) *hold at* $(\overline{x},y)$, *for all* $y\in S(\overline{x})$. *Then, there are real numbers* α_j *with* $j=1,\ldots,k$, β_i^s *with* $i=1,\ldots,p$, $s=1,\ldots,n+1$, v_s *with* $s=1,\ldots,n+1$ *and vectors* $u_s\in\mathbb{R}^n$, $y_s\in S(\overline{x})$ *with* $s=1,\ldots,n+1$ *such that*:

$$\sum_{s=1}^{n+1}v_s u_s\in\sum_{j=1}^{k}\alpha_j\partial G_j(\overline{x}), \tag{3.8}$$

$$\forall s=1,\ldots,n+1,\quad (u_s,0)\in\partial(-f)(\overline{x},y_s)+\sum_{i=1}^{p}\beta_i^s\partial g_i(\overline{x},y_s), \tag{3.9}$$

$$\forall s=1,\ldots,n+1,\quad i=1,\ldots,p,\ \beta_i^s\geq 0,\ \beta_i^s g_i(\overline{x},y_s)=0, \tag{3.10}$$

$$\forall j=1,\ldots,k,\quad \alpha_j\geq 0,\ \alpha_j G_j(\overline{x})=0, \tag{3.11}$$

$$\forall s=1,\ldots,n+1,\ v_s\geq 0,\ \sum_{s=1}^{n+1}v_s=1. \tag{3.12}$$

Proof Since all the functions involved in (P) are assumed to be local Lipschitz continuous, then combining the inner semicompactness of S around $\overline{x}$ and the fulfilment of CQs (3.2) and (3.3) at $(\bar{x}, y)$, for all $y \in S(\bar{x})$, it follows from [11, Theorem 5.2(ii)] (see also [10, Corollary 4.43]) that the value function φ_o is also Lipschitz continuous around $\bar{x}$. Hence, applying [10, Proposition 5.3] we have the following compact form of the optimality conditions of (P)

$$0 \in \partial\varphi(\bar{x}) + N_X(\bar{x}). \tag{3.13}$$

Considering the fact that $\varphi = -\varphi_o$ on X, the latter inclusion implies the following one

$$0 \in -\text{co}\,\partial\varphi_o(\bar{x}) + N_X(\bar{x}) \tag{3.14}$$

taking into account the convex property (2.5) which ensures that

$$\partial(-\varphi_o)(\bar{x}) \subseteq \text{co}\,\partial(-\varphi_o)(\bar{x}) = -\text{co}\,\partial\varphi_o(\bar{x}).$$

Let us now recall that from the analogue of Theorem 8 in [13] (see also [10, Corollary 4.36]), one has an upper estimate of the basic subdifferential of φ_o as

$$\partial\varphi_o(\bar{x}) \subseteq \bigcup_{y \in S(\bar{x})} \left\{ u | (u, 0) \in \partial(-f)(\bar{x}, y) + \sum_{i=1}^{p} \beta_i \partial g_i(\bar{x}, y) \right.$$
$$\left. \text{for } i = 1, \ldots, p,\ \beta_i \geq 0,\ \beta_i g_i(\bar{x}, y) = 0 \right\}, \tag{3.15}$$

considering the inner semicompactness of S at $\bar{x}$ and the satisfaction of CQ (3.2) at $(\bar{x}, y)$, for all $y \in S(\bar{x})$.

Applying Carathéodory's theorem, it follows from (3.15) that $u \in \text{co}\,\partial\varphi_o(\bar{x})$ implies the existence of β_i^s with $i = 1, \ldots, p$, $s = 1, \ldots, n+1$, v_s with $s = 1, \ldots, n+1$ and vectors $u_s \in \mathbb{R}^n$, $y_s \in S(\bar{x})$ with $s = 1, \ldots, n+1$ such that (3.9), (3.10), (3.12) and

$$u = \sum_{s=1}^{n+1} v_s u_s.$$

Combining the latter with (3.14) while noting the fact that

$$N_X(\bar{x}) \subseteq \bigcup \left\{ \sum_{j=1}^{k} \alpha_j \partial G_j(\bar{x}) | \text{ for } j = 1, \ldots, k,\ \alpha_j \geq 0,\ \alpha_j G_j(\bar{x}) = 0 \right\} \tag{3.16}$$

(under the upper level regularity), we have (3.8) and (3.11), which concludes the proof. ■

If instead of the inner semicompactness of S in the above theorem, one assumes the stronger inner semicontinuity, one obtains conditions for (P) that would appear to be very sharp in some cases, as it will be obvious in the subsequent sections. In the next result, one would also impose the following stronger nonsmooth extension of the MFCQ in terms of the Clarke subdifferential constructions:

$$\left[0 \in \sum_{i=1}^{p} \beta_i \bar{\partial} g_i(\bar{x}, \bar{y}),\ \beta_i \geq 0, \beta_i g_i(\bar{x}, \bar{y}) = 0, i = 1, \ldots, p \right] \Longrightarrow \beta_i = 0,\ i = 1, \ldots, p. \tag{3.17}$$

Considering the fact that $\partial\psi \subseteq \bar{\partial}\psi$ for a locally Lipschitz continuous function ψ, it is clear that the latter condition implies (3.2).

Theorem 3.2 (Lower subdifferential optimality conditions in the nonsmooth case under the inner semicontinuity of S) *Let $\bar{x}$ be an upper level regular local optimal solution for* (P), *where the functions f, g and G are all locally Lipschitz continuous. Furthermore, assume that S is inner semicontinuous around* $(\bar{x}, \bar{y})$ *and let CQs* (3.17) *and* (3.3) *hold at* $(\bar{x}, \bar{y})$. *Then, there are real numbers α_j with $j = 1, \ldots, k$, β_i with $i = 1, \ldots, p$ and a vector $u \in \mathbb{R}^n$ such that* (3.11) *and the following hold*:

$$u \in \sum_{j=k}^{k} \alpha_j \partial G_j(\bar{x}), \tag{3.18}$$

$$(u, 0) \in -\bar{\partial} f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \beta_i \bar{\partial} g_i(\bar{x}, \bar{y}), \tag{3.19}$$

$$\forall i = 1, \ldots, p, \; \beta_i \geq 0, \; \beta_i g_i(\bar{x}, \bar{y}) = 0. \tag{3.20}$$

Proof The proof follows on the path of that of Theorem 3.1. To be more precise, the local Lipschitz continuity of φ_o (that is of φ) is also obtained from [11, Theorem 5.2(i)] under (3.3), (3.17) and the inner semicontinuity of S around $(\bar{x}, \bar{y})$. Furthermore, considering equality (2.4), it follows from (3.14) that

$$0 \in -\bar{\partial} \varphi_o(\bar{x}) + N_X(\bar{x}). \tag{3.21}$$

Theorem 4.8 in [12] gives the following upper estimate for the Clarke subdifferential of φ_o under the inner semicontinuity of S and CQ (3.17):

$$\bar{\partial} \varphi_o(\bar{x}) \subseteq \left\{ u | (u, 0) \in -\bar{\partial} f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \beta_i \bar{\partial} g_i(\bar{x}, \bar{y}) \right. \\ \left. \text{for } i = 1, \ldots, p, \; \beta_i \geq 0, \; \beta_i g_i(\bar{x}, \bar{y}) = 0 \right\}, \tag{3.22}$$

where $\bar{\partial}$ stands for the Clarke subdifferential (see Section 2) of the function in question. Hence, combining (3.21), (3.22) and inclusion (3.16), we have the result. ■

Remark 3.1 (Inner semicompactness/inner semicontinuity) As noted in the above results, the inner semicompactness and inner semicontinuity, respectively, have played a major role. For the inner semicompactness, as mentioned in Section 2, it is automatically satisfied if S is nonempty and uniformly bounded (2.8), which is a weak requirement. As for the inner semicontinuity, it is obtained if S is Lipschitz-like around the point in question. Conditions ensuring that the solution set-valued mapping of an optimization problem is Lipschitz-like are developed in [5] under various settings. This condition also automatically holds at $(\bar{x}, \bar{y})$ provided S is closed-graph and inner semicompact at $\bar{x}$ with $S(\bar{x}) = \{\bar{y}\}$ or if S is the solution set-valued mapping of a parametric linear program with additive right-hand-side perturbations. Further details and references on the inner semicontinuity of a solution set-valued mapping can be found in [3, Remark 3.2].

Remark 3.2 (Convex combinations in necessary optimality conditions) If the functions $-f$ and g_i, $i = 1, \ldots, p$ are assumed to be lower/subdifferentially/Clarke regular (2.6) in Theorem 3.2, then the convex combination of subgradients of these functions do not appear as it is the case in the optimality conditions of Theorem 3.1. This allows getting optimality conditions, in terms of the basic lower subdifferential, of the KKT-type in the usual sense, as no number will be attached to the generalized gradient of the objective function f of the minmax program. This will be more clear in Theorem 4.1 and Corollary 5.2 when we consider special lower regularity cases for $-f$ and g_i, $i = 1, \ldots, p$, that is, the full convexity and continuous differentiability, respectively. To the best knowledge of the author, such optimality conditions have not been obtained before for the minmax programming problem. It may also be worth mentioning that the optimality conditions in Theorem 3.1 and Theorem 3.2 coincide if in the former result, the multiplier β is unique (see inclusion (3.15)), one has $S(\overline{x}) = \{\overline{y}\}$ and the functions $-f$ and g_i, $i = 1, \ldots, p$ are lower regular at $(\overline{x}, \overline{y})$.

Remark 3.3 (On the CQs used in Theorems 3.1 and 3.2) In Theorem 3.1 (resp. Theorem 3.2), CQs (3.2) and (3.3) (resp. (3.3) and (3.17)) could be replaced by the single CQ (3.4) (resp. CQ (3.4) where $\overline{\partial}$ substitutes ∂). Also of interest, let us mention that in Theorem 3.1, CQ (3.2) at $(\overline{x}, \overline{y})$ can be replaced by the weaker calmness of the set-valued mapping $v \rightrightarrows \{(x, y) | g(x, y) + v \leq 0\}$ at $(0, \overline{x}, \overline{y})$, which is automatically satisfied if the functions g_i, $i = 1, \ldots, p$ are affine linear. The latter statement is valid for the upper level regularity, used in both theorems, as well.

Remark 3.4 (On the CQ (3.3)) It is worth recalling that the combination of CQs (3.2) and (3.3), including of course the inner semicompactness or semicontinuity of S, is mostly to ensure the Lipschitz continuity of the value function φ (1.1). Otherwise, if the latter is not satisfied, we face two difficulties: (i) φ is more likely to be a upper semicontinuous function, which may not be a good news for the application of the basic lower subdifferential, as it is usually required that the objective function of the minimization problem be lower semicontinuous, cf. [10, Proposition 5.3]. (ii) As mentioned in Section 2, one is sure that the basic lower subdifferential of φ is nonempty (and compact) if φ is locally Lipschitz continuous. Otherwise, condition (3.13) may make no sense.

We now come to our choice to consider conditions (3.2) and (3.3) simultaneously, in order to secure the aforementioned Lipschitz continuity of φ (1.1). In fact, most often (especially in smooth case), one would consider only CQ (3.4), which is strong enough to ensure the satisfaction of both (3.2) and (3.3), cf. Remark 3.3. But it seems more strategic to consider the combination of the latter conditions. As a matter of fact, in our minmax program (P), if we replace the inner constraint $g(x, y) \leq 0$ by the inclusion $y \in \Psi(x) := \arg\min_y \{f_o(x, y) | \, g_o(x, y)\}$, conditions (3.2) and (3.4) do not hold, cf. [6], whereas CQ (3.3) can be satisfied, see [5]. As it was the case in the latter work, the fulfilment of CQ (3.3) while CQ (3.2) and (3.4) fail gives room to replace (3.2) by a weaker condition, for example, the calmness of a certain set-valued mapping (see Remark 3.3), thus helping to still secure the Lipschitz continuity of φ.

Remark 3.5 (Operator constraints in the minmax programming problem (P)) In Theorem 3.1 and Theorem 3.2, only simple inequality constraints have been considered for the outer problem ($G(x) \leq 0$) and for the inner problem ($g(x, y) \leq 0$). However, these results and most subsequent ones remain valid if the more general *operator constraint* structure $z \in \Omega \cap \psi^{-1}(\Lambda)$ (with ψ being a local Lipschitz continuous function and Λ a closed set while z corresponds to x or (x, y)) is in consideration. One simply has to adjust the CQs and optimality conditions accordingly. In particular, CQ (3.2) would take the form

$$\left[0 \in \partial\langle u, \psi\rangle(\bar{z}) + N_\Omega(\bar{z}),\ u \in N_\Lambda(\psi(\bar{z}))\right] \Longrightarrow u = 0.$$

The constraint structures treated in this article are particular operator constraints with $\psi := G$, $\Lambda := \mathbb{R}^k_-$, $\Omega := \mathbb{R}^n$ for the outer constraint and $\psi := g$, $\Lambda := \mathbb{R}^p_-$, $\Omega := \mathbb{R}^n \times \mathbb{R}^m$ for the inner one. The case where $\Omega := \operatorname{gph}\Psi$ (with $\Psi(x) := \arg\min_y\{f_o(x,y) |\, g_o(x,y)\}$) in the inner constraint is considered in [4]. Further details on how to handle such constraints can be found in the book by Mordukhovich [10] or Rockafellar and Wets [14].

To conclude this section, we consider the following so-called static or nonparametric minmax problem intensively investigated in the literature, see e.g. [15] and references therein and more recently [7,16]:

$$\min_{x\in X}\max_{y\in Y} f(x,y). \tag{3.23}$$

Here, X is defined as in (1.1) whereas Y is simply albeit a very general subset of $\mathbb{R}^m$. Apart of the Lipschitz continuity of the functions G and f, it will be required on the inner problem only that Y be a bounded set. In the next lemma, we first provide the relevant variational properties of the value function

$$\varphi_o(x) := \min\{-f(x,y) |\, y \in Y\}.$$

The projection mapping from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ denoted by $\operatorname{proj}_{\mathbb{R}^n}$ is defined for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ by $\operatorname{proj}_{\mathbb{R}^n}(x,y) = x$.

Lemma 3.3 *Assume that f is locally Lipschitz continuous and Y is a bounded set. Then, the value function φ_o is locally Lipschitz continuous and an estimate of its basic subdifferential is obtained as*

$$\partial\varphi_o(\bar{x}) \subseteq \cup\{\operatorname{proj}_{\mathbb{R}^n}\partial(-f)(\bar{x},y) |\, y \in S(\bar{x})\}. \tag{3.24}$$

If in addition, $-f$ is fully convex (resp. continuously differentiable), then one has

$$\partial\varphi_o(\bar{x}) \subseteq \cup\{\partial_x(-f)(\bar{x},y) |\, y \in S(\bar{x})\} (\text{resp. } \partial\varphi_o(\bar{x}) \subseteq \{-\nabla_x f(\bar{x},y) |\, y \in S(\bar{x})\}). \tag{3.25}$$

Proof The set Y being bounded, the set-valued mapping S is uniformly bounded (2.8). Moreover, the set-valued mapping $K(x) := Y$ is constant, and by the definition of the coderivative (2.10), one has $D^*K(x,y)(0) = \{0\}$, for all $(x,y) \in \mathbb{R}^n \times Y =: \operatorname{gph} K$, considering that we always have $0 \in N_Y(y)$. Therefore, by [11, Theorem 5.2], one has the local Lipschitz continuity of φ_o and by [13, Theorem 7], inclusion (3.24) is obtained. The inclusions in (3.25) are trivial consequences of (3.24). ■

Theorem 3.4 (Lower subdifferential optimality conditions in the nonsmooth case with nonparametric inner constraint) *Let $\bar{x}$ be an upper level regular local optimal solution for problem* (3.23), *where the functions f and G are all locally Lipschitz continuous and the set Y is bounded. Then, there are real numbers α_j with $j = 1, \ldots, k$, v_s with $s = 1, \ldots, n+1$ and vectors $u_s \in \mathbb{R}^n$, $y_s \in S(\bar{x})$ with $s = 1, \ldots, n+1$ such that* (3.8), (3.11) *and* (3.12) *are satisfied with*

$$u_s \in \operatorname{proj}_{\mathbb{R}^n}\partial(-f)(\bar{x}, y_s). \tag{3.26}$$

Proof It follows trivially by combining inclusion (3.14) of the proof of Theorem 3.1 and the upper estimate of $\partial\varphi_o(\bar{x})$ (3.24) in the previous lemma. ∎

Remark 3.6 These conditions are close to those obtained by Dhara and Mehra [7]. However, in Theorem 4.3 of the latter paper, it is not required that the y_ss be elements of the solution set $S(\bar{x})$. This makes them weaker than those obtained in Theorem 3.4. It should however be mentioned that the limiting dual objects used in [7] are limiting proximal sudifferentials and normal cones.

4. Minmax programs with convex and linear structures

In this section, we are interested in the situation where at least all the functions involved in the inner problem, that is $-f$ and g_i, $i=1,\dots,p$, are fully convex, which means convex in (x,y). Hence, the following Slater CQ will be more appropriate here: the Slater CQ will be said to hold for the system of inequalities $g(x,y)\leq 0$, if there exist $(\tilde{x},\tilde{y})$ satisfying

$$g_i(\tilde{x},\tilde{y}) < 0, \quad \text{whenever } i = 1,\dots,p. \tag{4.1}$$

The convexity of the components of G are not of a particular interest here since this does not make any significant impact on the optimality conditions. However, if for $i=1,\dots,p$, G_i is convex, the upper level regularity could be replaced by its Slater counterpart.

Theorem 4.1 (Lower subdifferential optimality conditions in the nonsmooth and fully convex case) *Let $\bar{x}$ be an upper level regular local optimal solution for* (*P*), *where G is locally Lipschitz continuous and $-f$, g_i, $i=1,\dots,p$ are all convex in (x,y). Furthermore, assume that CQ* (4.1) *holds and φ is finite near $\bar{x}$. Then, for $\bar{y}\in S(\bar{x})$, there are real numbers α_j with $j=1,\dots,k$ and β_i with $i=1,\dots,p$, such that* (3.11), (3.20) *and the following hold*:

$$0\in -\partial_x(-f)(\bar{x},\bar{y}) - \sum_{i=1}^{p}\beta_i\partial_x g_i(\bar{x},\bar{y}) + \sum_{j=k}^{k}\alpha_j\partial G_j(\bar{x}), \tag{4.2}$$

$$0\in -\partial_y(-f)(\bar{x},\bar{y}) - \sum_{i=1}^{p}\beta_i\partial_y g_i(\bar{x},\bar{y}). \tag{4.3}$$

Proof It is well-known that since the functions $-f$ and g are convex in (x,y), the value function φ_o is also a convex function, see e.g. [16]. Furthermore, φ_o is Lipschitz continuous around $\bar{x}$ given that it is finite around this point. Hence, the optimality condition (3.14) reduces to

$$0\in -\partial\varphi_o(\bar{x}) + N_X(\bar{x}). \tag{4.4}$$

with $\partial\varphi_o$ also denoting the subdifferential of φ_o in the sense of convex analysis. Therefore, considering (2.3) and the fact that $(\bar{x},\bar{y})\in \operatorname{gph} S$, $u\in\partial\varphi_o(\bar{x})$ implies

$$-f(x,y) - \langle u,x\rangle \geq -f(\bar{x},\bar{y}) - \langle u,\bar{x}\rangle, \quad \forall (x,y)\colon g(x,y)\leq 0,$$

which means that $(\bar{x}, \bar{y})$ is an optimal solution of the problem

$$\min_{x,y}\{-f(x,y) - \langle u, x\rangle \,|\, g(x,y) \le 0\}.$$

Hence, applying the Lagrange multiplier rule for nonsmooth optimization problems (see e.g. [10]) to the latter problem, it follows, under CQ (4.1), that there exists β such that (3.20) and

$$(u, 0) \in \partial(-f)(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \beta_i \partial g_i(\bar{x}, \bar{y}). \tag{4.5}$$

In summary, an upper bound of the basic subdifferential of φ_o can be estimated as

$$\partial\varphi_o(\bar{x}) \subseteq \bigcup_{\beta\in\Lambda(\bar{x},\bar{y})} \left\{ \partial_x(-f)(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \beta_i \partial_x g_i(\bar{x}, \bar{y}) \right\}, \tag{4.6}$$

taking into account that since $-f$ and g are fully convex, one has the inclusions $\partial(-f)(\bar{x}, \bar{y}) \subseteq \partial_x(-f)(\bar{x}, \bar{y}) \times \partial_y(-f)(\bar{x}, \bar{y})$ and $\partial g(\bar{x}, \bar{y}) \subseteq \partial_x g(\bar{x}, \bar{y}) \times \partial_y g(\bar{x}, \bar{y})$. In (4.6), $\Lambda(\bar{x}, \bar{y})$ denotes the set of Lagrange multipliers

$$\Lambda(\bar{x}, \bar{y}) := \left\{ \beta | 0 \in \partial_y(-f)(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \beta_i \partial_y g_i(\bar{x}, \bar{y}) \right.$$
$$\left. \beta_i \ge 0,\ \beta_i g_i(\bar{x}, \bar{y}) = 0,\ i = 1, \ldots, p \right\}.$$

Combining (4.4) and (4.6), one has the result. ∎

With the full convexity of $-f$, g_i, $i = 1, \ldots, p$, the optimality conditions of Theorem 3.2 and Theorem 4.1 are in fact the same. It should however be mentioned that there is no relation between convexity and inner semicontinuity, cf. [3, Remark 3.2] for an illustration. It is now much clear that we have necessary optimality conditions for the minmax program without a convex combination of the generalized gradient of the objective function f. Hence, we have KKT-type conditions in the usual sense.

We terminate this section by considering the special case of the above theorem, where all the functions involved in (P) are affine linear.

Corollary 4.2 (Lower subdifferential optimality conditions for the minmax problem with a linear structure): *Let $\bar{x}$ be a local optimal solution of (P) in the case where $G(x) := Dx + d$, $f(x,y) := a^\top x + b^\top y$ and $g(x,y) := Ax + By + c$, with the sizes of the vectors a, b, c, d and the matrices A, B and D chosen accordingly. Then, for $\bar{y} \in S(\bar{x})$, there are real numbers α_j with $j = 1, \ldots, k$ and β_i with $i = 1, \ldots, p$, such that:*

$$a + \sum_{i=1}^{p} \beta_i A_i + \sum_{j=1}^{k} \alpha_j D_j = 0,$$
$$b + \sum_{i=1}^{p} \beta_i B_i = 0,$$
$$\text{for } j = 1, \ldots, k,\ \alpha_j \ge 0,\ \alpha_j(D_j\bar{x} + d_j) = 0,$$
$$\text{for } i = 1, \ldots, p,\ \beta_i \ge 0,\ \beta_i(A_i\bar{x} + B_i\bar{y} + c_i) = 0.$$

Proof Simply note that the functions G, f and g are all fully convex. Furthermore, with the affine linearity of the function g, the inclusion in (4.6) is automatically satisfied. For the fulfilment of the CQs, see Remark 3.3. ■

5. Minmax programs with smooth initial data

We consider problem (P) in the case where all the functions f, g_i, $i=1,\dots,p$ and G_j, $j=1,\dots,k$ are continuously differentiable. Hence, the smooth counterpart of the upper level regularity (3.1) takes the form

$$\left[\sum_{j=1}^{k}\alpha_j\nabla G_j(\overline{x})=0,\ \alpha_j\geq 0,\ \alpha_jG_j(\overline{x})=0,\ j=1,\dots,k\right]\Longrightarrow \alpha_j=0, j=1,\dots,k, \tag{5.1}$$

whereas for the inner/lower level constraints, g_i, $i=1,\dots,p$, CQ (3.4) takes the form

$$\left[\sum_{i=1}^{p}\beta_i\nabla_y g_i(\overline{x},\overline{y})=0,\ \beta_i\geq 0,\ \beta_i g_i(\overline{x},\overline{y})=0,\ i=1,\dots,p\right]\Longrightarrow \beta_i=0,\ i=1,\dots,p. \tag{5.2}$$

We do not consider the smooth counterparts of CQs (3.2) and (3.3). They can well be introduced and handled as in Section 3 while considering the important point made in Remark 3.4. The next result is a consequence of Theorem 3.1.

Corollary 5.1 (Lower subdifferential optimality conditions in the smooth case under inner semicompactness) *Let $\overline{x}$ be an upper level regular* (5.1) *local optimal solution for* (*P*), *where f, g_i, $i=1,\dots,p$ and G_j, $j=1,\dots,k$ are all continuously differentiable*. *Furthermore*, *assume that S is inner semicompact around $\overline{x}$ and let the lower level regularity* (5.2) *holds at* $(\overline{x},y)$, *for all* $y\in S(\overline{x})$. *Then*, *there are real numbers α_j with $j=1,\dots,k$, β_i^s with $i=1,\dots,p$; $s=1,\dots,n+1$, v_s with $s=1,\dots,n+1$ and vectors $y_s\in S(\overline{x})$ with $s=1,\dots,n+1$ such that* (3.10)–(3.12) *and*

$$\sum_{s=1}^{n+1}v_s\left(\nabla_x f(\overline{x},y_s)-\sum_{i=1}^{p}\beta_i^s\nabla_x g_i(\overline{x},y_s)\right)+\sum_{j=1}^{k}\alpha_j\nabla G(\overline{x})=0, \tag{5.3}$$

$$\forall s=1,\dots,n+1,\ \nabla_y f(\overline{x},y_s)-\sum_{i=1}^{p}\beta_i^s\nabla_y g_i(\overline{x},y_s)=0. \tag{5.4}$$

In case S (3.5) is instead inner semicontinuous, one has the following corollary of Theorem 3.2.

Corollary 5.2 (Lower subdifferential optimality conditions in the smooth case under inner semicontinuity) *Let $\overline{x}$ be an upper level regular* (5.1) *local optimal solution for* (*P*), *where f, g_i, $i=1,\dots,p$ and G_j, $j=1,\dots,k$ are all continuously differentiable*. *Furthermore*, *assume that S is inner semicontinuous around* $(\overline{x},\overline{y})$ *and let the lower level regularity* (5.2) *hold at* $(\overline{x},\overline{y})$. *Then*, *there are real numbers α_j with*

$j=1,\dots,k$ *and* β_i *with* $i=1,\dots,p$ *such that* (3.11) *and* (3.20) *and*

$$\nabla_x f(\overline{x},\overline{y}) - \sum_{i=1}^{p} \beta_i \nabla_x g_i(\overline{x},\overline{y}) + \sum_{j=1}^{k} \alpha_j \nabla G(\overline{x}) = 0, \tag{5.5}$$

$$\nabla_y f(\overline{x},\overline{y}) - \sum_{i=1}^{p} \beta_i \nabla_y g_i(\overline{x},\overline{y}) = 0. \tag{5.6}$$

If we replace the inner semicontinuity of S in this corollary by the full convexity of $-f$ and g_i, $i=1,\dots,p$, one would get the same optimality conditions by means of Theorem 4.1. But as mentioned in the previous section, both assumptions are not related.

Corollary 5.3 (Lower subdifferential optimality conditions in the smooth case with nonparametric inner constraints) *Let* $\overline{x}$ *be an upper level regular local optimal solution for problem* (3.23), *where the functions* f *and* G *are all continuously differentiable and the set* Y *is bounded. Then, there are real numbers* α_j *with* $j=1,\dots,k$, *and* v_s *with* $s=1,\dots,n+1$ *and vectors* $y_s \in S(\overline{x})$ *with* $s=1,\dots,n+1$ *such that* (3.11), (3.12) *and*

$$\sum_{s=1}^{n+1} v_s \nabla_x f(\overline{x}, y_s) + \sum_{j=1}^{k} \alpha_j \nabla G(\overline{x}) = 0. \tag{5.7}$$

These conditions are close to those of [15, Theorem 1]. But, if we consider the full convexity on the data of the inner problem or the inner semicontinuity of S, as intensively discussed above, the convex combination on the gradient of f is not necessary and we get KKT-type conditions.

To end this section, we derive *upper subdifferential* optimality conditions for the minmax program (P). For the remainder, a feasible point $\overline{x}$ of (P) will be said to satisfy upper subdifferential optimality conditions if

$$-\widehat{\partial}^{+}\varphi(\overline{x}) \subseteq \widehat{N}_X(\overline{x}), \tag{5.8}$$

where $\widehat{\partial}^{+}\varphi(\overline{x})$ (resp. $\widehat{N}_X(\overline{x})$) denotes the upper Fréchet subdifferential of φ (resp. the Fréchet normal cone of X) at $\overline{x}$, cf. [9]. According to Mordukhovich [9], these conditions are most suitable for a problem of minimizing a concave function. Let us note that if the functions $-f$ and g_i, $i=1,\dots,p$ are fully convex, the minmax program (P) falls in this category of problems. Hence, in the next theorem, we derive a different kind of optimality conditions for (P) in the case where $-f$ and g_i, $i=1,\dots,p$ are continuously differentiable and convex in (x,y).

Theorem 5.4 (Upper subdifferential optimality conditions for the smooth minmax program) *Let* $\overline{x}$ *be an upper level regular* (5.1) *local optimal solution for* (*P*), *where* f, g_i, $i=1,\dots,p$ *and* G_j, $j=1,\dots,k$ *are all continuously differentiable and* $-f$ *and* g_i, $i=1,\dots,p$ *are also convex in* (x,y). *Furthermore, let* φ_o *be finite near* $\overline{x}$ *and assume that* S *is inner semicompact at* $\overline{x}$ *and CQ* (5.2) *holds at* $(\overline{x},\overline{y})$. *Then, one has*

$$\forall \beta \in \mathbb{R}^p \text{ satisfying } (3.20), (5.6), \exists \alpha \in \mathbb{R}^k \text{ verifying } (3.11) \text{ such that } (5.5). \tag{5.9}$$

Proof Since φ is finite at $\overline{x}$, one has from [9, Theorem 3.1(i)] that

$$-\widehat{\partial}^{+}\varphi(\overline{x}) \subseteq N_X(\overline{x}). \tag{5.10}$$

As mentioned above, φ is concave, since $-f$ and g_i, $i=1,\ldots,p$ are all fully convex. Hence, φ is upper regular. Moreover, this function is finite around $\overline{x}$. Hence, Lipschitz continuous near the latter point. Therefore, one has from [9] that

$$\widehat{\partial}^{+}\varphi(\overline{x}) = \overline{\partial}\varphi(\overline{x}). \tag{5.11}$$

For the remainder of the reader, $\overline{\partial}\varphi$ stands for the Clarke subdifferential of φ (2.4). On the other hand, one has the following sequence of equalities thanks to the plus/minus symmetry enjoyed by the latter subdifferential concept:

$$\overline{\partial}\varphi(\overline{x}) = -\overline{\partial}(-\varphi)(\overline{x}) = -\partial\varphi_o(\overline{x}). \tag{5.12}$$

More precisely, $\partial\varphi_o$ is the subdifferential in the sense of convex analysis. Making the corresponding substitution in inclusion (5.10), one has

$$\partial\varphi_o(\overline{x}) \subseteq N_X(\overline{x}). \tag{5.13}$$

We know from [16, Theorem 6.6.7] that under the assumptions of the theorem, one has

$$\partial\varphi_o(\overline{x}) = \bigcup_{\beta\in\Lambda(\overline{x},\overline{y})} \left\{-\nabla_x f(\overline{x},\overline{y}) + \sum_{i=1}^{p} \beta_i \nabla_x g_i(\overline{x},\overline{y})\right\}, \tag{5.14}$$

where the set of Lagrange multipliers $\Lambda(\overline{x},\overline{y})$ is obtained as

$$\begin{aligned}\Lambda(\overline{x},\overline{y}) := \Bigg\{\beta | -\nabla_y f(\overline{x},\overline{y}) + \sum_{i=1}^{p} \beta_i \nabla_y g_i(\overline{x},\overline{y}) = 0 \\ \beta_i \geq 0,\ \beta_i g_i(\overline{x},\overline{y}) = 0,\ i = 1,\ldots,p\Bigg\}.\end{aligned} \tag{5.15}$$

Combining (5.13) and (5.14) while taking into account the fact that

$$N_X(\overline{x}) = \left\{\sum_{j=1}^{k} \alpha_j \nabla G_j(\overline{x}) | \text{ for } j = 1,\ldots,k,\ \alpha_j \geq 0,\ \alpha_j G_j(\overline{x}) = 0\right\}$$

(under the upper level regularity (5.1)), one has the result. ■

The first observation to make about these optimality conditions is that we have the quantifier 'for all' attached to the multiplier β instead of 'there exists' in all the other optimality conditions previously obtained in this article. This makes the optimality conditions of Theorem 5.4 stronger than those of Corollary 5.2. The optimality conditions of both results coincide if β is unique. This can be ensured by replacing CQ (5.2) by the well-known linear independence CQ (LICQ).

The optimality conditions in Theorem 5.4 have the same structure as those derived by Ishizuka [8]. In particular, condition (5.9) coincide with the optimality conditions in Corollary 9.2.1 of his book with Shimizu and Bard [16], provided in the latter result, the solution set of the inner problem is a singleton at $\overline{x}$, that is, $\arg\max_y \{f(\overline{x},y) | g(\overline{x},y) \leq 0\} = \{\overline{y}\}$. It should however be mentioned that the approach used by Ishizuka [8] (see also [16, Chapter 9]) is completely different from the one we have used here, since their results are based on nonconvex generalizations of Farkas' alternative theorem.

Remark 5.1 (Upper/superdifferential regularity of the value function φ (1.1) without full convexity) Following the proof of Theorem 5.4, the full convexity of $-f$ and g_i, $i=1,\dots,p$ provided for two major things, that is, the Lipschitz continuity and upper/supperdifferential regularity of φ (1.1). The latter is in fact a main argument for the suggestion in [9] to consider upper subdifferential optimality conditions. The question now is how to get the upper regularity of φ beyond the full convexity of the functions $-f$ and g_i, $i=1,\dots,p$. If the latter functions are continuously differentiable and convex only in the inner variable y, the solution set-valued mapping S (3.5) is uniformly bounded at $\overline{x}$ and there exits $\widetilde{y}$ such that the Slater CQ (4.1) holds at $(\overline{x},\widetilde{y})$, then, φ_o is lower/subdifferentially regular at $\overline{x}$ if and only if

$$\overline{\partial}\varphi_o(\overline{x}) = \bigcap_{y\in S(\overline{x})}\left[\bigcup_{\beta\in\Lambda(\overline{x},\,y)}\left\{-\nabla_x f(\overline{x},\overline{y}) + \sum_{i=1}^{p}\beta_i\nabla_x g_i(\overline{x},\overline{y})\right\}\right],$$

where the multipliers set $\Lambda(\overline{x},y)$ is defined in (5.15), cf. Shimizu et al. [16, Theorem 6.5.5]. Hence, ensuring the upper regularity of φ (1.1).

It is also an obvious fact, that if a function is strictly differentiable, then it is upper regular. Furthermore, we know, see [10], that a locally Lipschitz continuous function is strictly differentiable if it has a single (basic) subgradient. Hence, combining the inner semicontinuity of S and the LICQ, one has the strict differentiability of the value function φ via the combination of [11, Theorem 5.2] and [13, Corollary 4]. It should however be mentioned that the strict differentiability of φ is not interesting for deriving the upper subdifferential optimality conditions of our minmax problem (P), as the latter conditions would coincide in this case with the lower subdifferential ones.

6. Conclusions

We have investigated upper and lower subdifferential optimality conditions for the minmax programming problem in various settings, that is, when the initial data is Lipschitz continuous, convex or continuously differentiable. It results that in the smooth static case, the lower subdifferential optimality conditions obtained here are close to those of [15] whereas in the smooth parametric case our upper conditions may be identical to those of [8,16]. Only simple inequality constraints have been considered for the outer problem ($G(x)\leq 0$) and for the inner problem ($g(x,y)\leq 0$). However, most of the results in this article remain valid if the more general *operator constraint* structure $z\in\Omega\cap\psi^{-1}(\Lambda)$ (with ψ being a local Lipschitz continuous function and Λ a closed set while z corresponds to x or (x,y)) is in consideration. One simply has to adjust the CQs and optimality conditions accordingly, cf. Remark 3.5.

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