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Continuous Optimization

An exact penalty on bilevel programs with linear vector optimization lower level

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ABSTRACT

We are interested in a class of linear bilevel programs where the upper level is a linear scalar optimization problem and the lower level is a linear multi-objective optimization problem. We approach this problem via an exact penalty method. Then, we propose an algorithm illustrated by numerical examples.

1. Introduction

The bilevel programming problem is an optimization problem in which the constraints are implicitly determined by another optimization problem. In other words, it is an hierarchical optimization problem consisting in two levels. At the upper level, the decision maker (leader) has to choose first a strategy x to minimize his objective function F, and the lower level decision maker (follower) has to select a strategy y that minimizes its own objective function f parameterized by x. Anticipating the reaction of the follower, the leader intends to find such values for its variables which together with the follower's reaction minimize its objective function.

The bilevel programming problem is very hard to solve due to its non convexity and the implicity of its feasible set. Even if all the functions are linear and the feasible sets are polyhedron, this problem remains non-convex. For this reason, this problem has received a large attention especially for the linear case, see for example Aboussoror and Mansouri (2005), Campelo et al. (2000). For an extensive bibliography the reader can refer to Dempe (2003), Vicente and Calamai (1994).

In this paper, we are concerned with a bilevel programming problem where the upper level is a scalar optimization problem and the lower level is a multi-objective optimization problem. This situation can be interpreted either as there is a follower that has several objectives or as there are many decision makers in the lower level. In this later case, the leader must take into consideration the reaction of all of these followers. If we make the intersection of the feasible set of the followers, we will have a multi-objective optimization problem in the lower level.

In a mathematical term, the problem is

$$(BLP)\quad \operatorname*{Min}_{\substack{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m\\ x\in X,\, y\in\mathscr{M}(x)}} F(x,y),$$

where $\mathscr{M}(x)$ is the set of the efficient or weakly efficient solutions of the multi-objective optimization problem

$$(MOP)\quad \operatorname*{Min}_{\substack{y\in\mathbb{R}^m\\ y\in Y(x)}} f(x,y),$$

with

$$F:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R},\; f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^p,\; X\subset\mathbb{R}^n,\quad\text{and}\quad Y(x)\subset\mathbb{R}^m,\quad \forall x\in X.$$

This kind of problem has been considered in the first time by Bonnel (2006), the author gave necessary optimality conditions to the problem (BLP) when the function f is convex, and he has applied the results to the case when f is linear. This problem has been considered also by Bonnel and Morgan (2006), the authors have treated the problem (BLP) in the general case, where the functions f and F are defined on Hausdorff topological space. They have used an exterior penalty method to solve the problem.

In a multi-objective programming problem, several objective functions have to be minimized simultaneously. Usually, no single point will minimize all of the several objective functions given at once. This is due to the fact that the space $\mathbb{R}^p, p\geqslant 2$, can not be ordered totally. Therefore, the concept of optimality has to be replaced by a weaker concept called efficiency or Pareto-optimality. For this reason, we have mentioned in the problem (BLP) that $\mathscr{M}(x)$ is the set of the efficient solutions or the set of the weakly efficient solutions of the problem (MOP). In this paper, we deal with weakly efficient solutions. We will give the definition of the efficiency in the next section.

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In the last decade, many authors like Benson (1984, 1983), Benson and Sayin (1994), Bolintineanu (1993), Bolintineanu and El maghri (1997), Gal (1977), Isermann (1977), Tamara and Miura (1977), were interested in the linear multi-objective optimization problem:

$$(LMOP)\ \operatorname*{Max}_{x\in X_1} C_1 x,$$

with

$$C_1\in\mathbb{R}^{p\times n},\quad X_1=\{x\in\mathbb{R}^n/A_1x\leqslant b_1,\ x\geqslant 0\},\quad A_1\in\mathbb{R}^{m\times n}\ \text{ and }\ b_1\in\mathbb{R}^m.$$

After the characterization of the efficient point, they were interested in solving the problem of minimization of a function $f_1:\mathbb{R}^n\to\mathbb{R}$ over the efficient set $\mathscr{E}$ of the problem ($LMOP$):

$$(S)\ \operatorname*{Min}_{x\in\mathscr{E}} f_1(x).$$

Note that the efficient set $\mathscr{E}$ is not convex in general, and hence (S) is a global optimization problem.

In this work, we are concerned with the following problem:

$$(P)\quad \operatorname*{Min}_{\substack{x,y\\ x\in X, x\geqslant 0\\ y\in\mathscr{E}(x)}} F(x,y),$$

where $\mathscr{E}(x)$ is the set of the weakly efficient solution of the problem

$$MOP(x)\quad \operatorname*{Min}_{\substack{y\in\mathbb{R}^m\\ Ax+By\leqslant b\\ y\geqslant 0}} Cy,$$

with F is a concave function on $\mathbb{R}^n\times\mathbb{R}^m, C\in\mathbb{R}^{r\times m},\ A\in\mathbb{R}^{p\times n}$, $B\in\mathbb{R}^{p\times m}$, $b\in\mathbb{R}^p, X$ is a closed subset of $\mathbb{R}^n$ and $^{\mathrm{T}}$ denotes the transposition. We will approach the problem (P) via an exact penalty method inspired from Bolintineanu and El maghri (1997), El maghri (1996). Finally, we will study the all linear case (the function F is linear) and we will propose an algorithm.

The outline of this paper is as follows: in Section 2, we give some preliminaries and establish a result on the existence of solutions of the problem (P). In Section 3, we present the penalty method and our main result. Finally, in Section 4, we study the linear case and we propose an algorithm illustrated by numerical examples.

2. Preliminaries and existence of solutions

It is not possible to find an absolute solution that would be optimal for all the objective functions simultaneously, because there is no natural ordering in the objective-valued space but only a partial order.

Set $X^+=\{x\in X/x\geqslant 0\}$ and for all $x\in X^+$, set

$$Y(x)=\{y\in\mathbb{R}^m/Ax+By\leqslant b,\ y\geqslant 0\}.$$

Definition 2.1. Let $x\in X^+$. A vector $\hat{y}\in Y(x)$ is called:

(i) Efficient solution if there is no vector $y\in Y(x)$ such that $Cy\leqslant C\hat{y}$ and $Cy\neq C\hat{y}$.
(ii) Weakly efficient solution to the problem $MOP(x)$ if there is no vector $y\in Y(x)$ such that $Cy<C\hat{y}$.

Remark 2.1. Remark that in the definition of the objective function of the second level, we have ignored a term of the form $E^{\mathrm{T}}x$, since for a given x, $E^{\mathrm{T}}x$ is a constant in the follower's problem $MOP(x)$.

Throughout the paper, we assume that the following assumptions are satisfied.

$$(H_1)\ \begin{cases}\text{(i)} & \text{For any } x\in X^+,\ Y(x)\neq\emptyset,\\ \text{(ii)} & \text{there exists a compact subset } Z \text{ of } \mathbb{R}^m, \text{such that}\\ & Y(x)\subset Z \text{ for all } x\in X^+.\end{cases}$$

(H_2) The set X^+ is a polytope.

The following theorem and lemma show that the problem (P) admits at least one solution.

Lemma 2.1. *Under the assumptions (H_1) and (H_2), $\mathscr{E}(x)\neq\emptyset$, and $\mathscr{E}(x)$ is closed.*

Theorem 2.1. *Suppose that the assumptions (H_1) and (H_2) are satisfied. Then, the problem (P) has at least one solution.*

Proof. By Lemma 2.1, $\mathscr{E}(x)$ is a closed subset of the compact set $Y(x)$, then $\mathscr{E}(x)$ is compact. The proof follows by using the theorem of Weierstrass.

3. Penalization of the problem (*P*)

It is well known that the principle of the penalty method consists in approaching the problem (P) by a sequence of problems (P_{μ_k}) called penalized problems. Under some hypotheses, every accumulation point of a sequence $(x_k)_{k\in\mathbb{N}}$ of solutions of the penalized problems (P_{μ_k}) is a solution of the problem (P).

For $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$, let the function $h(x,y)$ be the optimal value of the maximization problem

$$PL(x,y)\quad \operatorname*{Max}_{u,t}\{t: Cu+te\leqslant Cy,\ u\in Y(x),\ t\in\mathbb{R}\},$$

which is equivalent to the following maximization problem

$$\widetilde{PL}(x,y)\quad \operatorname*{Max}_{u\in Y(x)}\ \min_{j=1,\dots,r} C_jy-C_ju,$$

in the sense that if $(\hat{u},\hat{t})$ solves $PL(x,y)$, then $\hat{u}$ solves $\widetilde{PL}(x,y)$, $\hat{t}=h(x,y)$ and $h(x,y)$ is the optimal value of the problem $\widetilde{PL}(x,y)$. With $e=(1,\dots,1)^{\mathrm{T}}\in\mathbb{R}^r$ and C_j is the jth row of the matrix C. In the sequel, we give some properties of the marginal function h.

Remark 3.1. Under the assumption (H_1) the marginal function h is well defined. In fact, the function $\min_{j=1,\dots,r} C_jy-C_ju$, is continuous over the compact set $Y(x)$.

Lemma 3.1. *h is a penalty function of the problem (P). In other words, the following assertions are satisfied.*

(i) $\forall x\in X^+,\ y\in Y(x)$: $h(x,y)\geqslant 0$,
(ii) $\forall x\in X^+:\ \mathscr{E}(x)=\{y\in Y(x)/h(x,y)=0\}$.

Proof

(i) Let $x\in X^+$, $y\in Y(x)$. We have $CY+0.e\leqslant CY$, thus $(y,0)$ is a feasible solution of the problem $PL(x,y)$. Hence $h(x,y)\geqslant 0$.
(ii) Let $x\in X^+$ and $y\in\mathscr{E}(x)$, thus $y\in Y(x)$.

Suppose that $h(x,y)\neq 0$. According to the assertion (i), we have $h(x,y)>0$. Then there exists a feasible solution (u,t) of the problem $PL(x,y)$ such that $t>0$. Thus, $Cu-Cy\leqslant -te<0$, which contradicts the fact that $y\in\mathscr{E}(x)$. Inversely, let $y\in Y(x)$ such that $h(x,y)=0$, and suppose that $y\notin\mathscr{E}(x)$. Then there exists $u\in Y(x)$ such that $Cu-Cy<0$. Therefore, we can find $t>0$ such that

$Cu - Cy \leqslant -te < 0$. This contradicts $h(x, y) = 0$ (according to the problem $PL(x, y)$). Hence $y \in \mathscr{E}(x)$. □

Lemma 3.2. *h is a concave function over $\mathbb{R}^n \times \mathbb{R}^m$.*

Proof. Let $x, x' \in \mathbb{R}^n$ and $y, y' \in \mathbb{R}^m$. According to the definition of h, there exists $(u, u') \in Y(x) \times Y(x')$ such that

$$Cu + h(x,y)e \leqslant Cy \quad \text{and} \quad Cu' + h(x',y')e \leqslant Cy'.$$

For $\alpha \in [0, 1]$ we have

$$C(\alpha u + (1-\alpha)u') + (\alpha h(x,y) + (1-\alpha)h(x',y'))e \leqslant C(\alpha y + (1-\alpha)y').$$

It is easy to show that $\alpha u + (1 - \alpha)u' \in Y(\alpha x + (1 - \alpha)x')$. Then

$$(\alpha u + (1-\alpha)u', (\alpha h(x,y) + (1-\alpha)h(x',y'))$$

is a feasible solution of the problem $PL(\alpha x + (1 - \alpha)x', \alpha y + (1 - \alpha)y')$.
Hence

$$h(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y') \geqslant \alpha h(x,y) + (1-\alpha)h(x',y'),$$

i.e. h is a concave function over $\mathbb{R}^n \times \mathbb{R}^m$. □

For $\mu > 0$, let the penalized problem

$$(P_\mu) \ \underset{\substack{x,y \\ x \in X^+ \\ y \in Y(x)}}{\text{Min}} + F(x,y) + \mu h(x,y).$$

Remark 3.2. A point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ is a feasible solution of the problem (P_μ) if and only if y is a feasible solution of the problem of the lower level $MOP(x)$.

Theorem 3.1. *Let $\mu > 0$. Under the assumptions (H_1) and (H_2) the problem (P_μ) has at least one solution.*

Proof. According to the Lemma 3.2, $F + \mu h$ is a concave function. Then, the penalized problem (P_μ) is a concave minimization problem over a polytope (assumptions (H_1) and (H_2)). Therefore, it admits at least one solution which is a vertex of this polytope (Zwart (1974), Falk and Hoffman (1976)). □

Let $(\mu_k)_{k \in \mathbb{N}}$ be a positive strictly increasing sequence such that $\mu_k \nearrow +\infty$, (x_{μ_k}, y_{μ_k}) the optimal solution of the problem (P_{μ_k}) and denote $(x_k, y_k) = (x_{\mu_k}, y_{\mu_k})$.

Theorem 3.2. *Under the assumptions (H_1) and (H_2) we have:*

(1) *The sequence $(x_k, y_k)_{k \in \mathbb{N}}$ admits at least an accumulation point, and every accumulation point is a solution of the problem (P).*
(2) *The sequences $(F(x_k, y_k))_{k \in \mathbb{N}}$ and $(h(x_k, y_k))_{k \in \mathbb{N}}$ are respectively, increasing and decreasing. Besides, $\lim_k h(x_k, y_k) = 0$.*
(3) *$\lim_k F(x_k, y_k) = F^*$, and $\lim_k \mu_k h(x_k, y_k) = 0$, with F^* denoting the optimal value of the problem (P).*

Proof

(1) The sequence $(x_k, y_k) \in X^+ \times Y(x_k)$ which is a compact set (assumptions (H_1) and (H_2)). Then, it admits at least one accumulation point. Besides, the sequence $(\mu_k)_{k \in \mathbb{N}}$ is strictly increasing. Then, by the definition of the (x_k, y_k) and according to the Lemma 3.1, one has for all $k \in \mathbb{N}$

$$F(x_k, y_k) + \mu_k h(x_k, y_k) \leqslant F(x_{k+1}, y_{k+1}) + \mu_k h(x_{k+1}, y_{k+1}) \tag{3.1}$$

and

$$F(x_{k+1}, y_{k+1}) + \mu_k h(x_{k+1}, y_{k+1}) \leqslant F(x_{k+1}, y_{k+1}) + \mu_{k+1} h(x_{k+1}, y_{k+1}). \tag{3.2}$$

Denote by (x^*, y^*) a solution of the problem (P) and F^* its optimal value. We also have

$$F(x_k, y_k) \leqslant F(x_k, y_k) + \mu_k h(x_k, y_k) \leqslant F(x^*, y^*) + \mu_k h(x^*, y^*) = F(x^*, y^*) = F^*. \tag{3.3}$$

Let $(\bar{x}, \bar{y})$ an accumulation point of the sequence $(x_k, y_k)_{k \in \mathbb{N}}$ and let $(x_k, y_k)_{k \in K}$, $K \subset \mathbb{N}$, be a subsequence that converges to $(\bar{x}, \bar{y})$. Then, by the continuity of function F, and by inequality (3.1), we have

$$F(\bar{x}, \bar{y}) \leqslant F^*. \tag{3.4}$$

Now, let us show that $\bar{y} \in \mathscr{E}(\bar{x})$. In fact, it is easy to see that $\bar{y} \in Y(\bar{x})$. According to the Lemma 3.1, it is sufficient to verify that $h(\bar{x}, \bar{y}) = 0$. By the inequalities (3.1), (3.2) and (3.3), one deduces that the sequence $(F(x_k, y_k) + \mu_k h(x_k, y_k))_{k \in K}$ converges. Since the sequence $(F(x_k, y_k))_{k \in K}$ converges, it follows that the sequence $(\mu_k h(x_k, y_k))_{k \in K}$ also converges. Therefore, the continuity of the function h, which is a concave function over $\mathbb{R}^n \times \mathbb{R}^m$, implies that $\lim_k h(x_k, y_k) = h(\bar{x}, \bar{y})$. Suppose that $h(\bar{x}, \bar{y}) \neq 0$. Then, $\lim_k \mu_k h(x_k, y_k) = +\infty$, which leads to a contradiction. Hence, $h(x_k, y_k) = 0$. Since $\bar{y} \in \mathscr{E}(\bar{x})$ and using the inequality (3.4), it follows that $(\bar{x}, \bar{y})$ is a solution of the problem (P).

(2) By the definition of x_{k+1}, one has

$$F(x_{k+1}, y_{k+1}) + \mu_{k+1} h(x_{k+1}, y_{k+1}) \leqslant F(x_k, y_k) + \mu_{k+1} h(x_k, y_k). \tag{3.5}$$

In the inequality (3.1), it is easy to see that if $(h(x_k, y_k))_{k \in \mathbb{N}}$ is a decreasing sequence, then $(F(x_k, y_k))_{k \in \mathbb{N}}$ is an increasing sequence. Besides, by adding the inequalities (3.1) and (3.5), we obtain

$$(\mu_{k+1} - \mu_k)(h(x_{k+1}, y_{k+1}) - h(x_k, y_k)) \leqslant 0.$$

Then, the result follows from the fact that the sequence $(\mu_k)_{k \in \mathbb{N}}$ is strictly increasing. Let us show now that $\lim_k h(x_k, y_k) = 0$. Since the sequence $(h(x_k, y_k))_k$ is bounded from below by 0 (Lemma 3.2) and is decreasing, then it is convergent. Set $\bar{h}$ its limit and let us consider an accumulation point $(\bar{x}, \bar{y})$ of the sequence $(x_k, y_k)_{k \in \mathbb{N}}$. By the assertion (1), $(\bar{x}, \bar{y})$ is an optimal solution of the problem (P). Since h is continuous then, $\bar{h} = h(\bar{x})$ and also $\bar{h} = 0$.

(3) According to the inequality (3.3), the sequence $(F(x_k, y_k))_{k \in \mathbb{N}}$ is bounded. Then, by the assertions (1) and (2) it converges to F^*. Using again (3.3), one obtains $\lim_k \mu_k h(x_k, y_k) = 0$. □

In the sequel, denote $\mathscr{A}_\mu$ the set of solutions of the problem (P_μ), and $\mathscr{A}$ the set of solutions of the problem (P). Recall that $\mathscr{A} \neq \emptyset$ (Theorem 3.2) and for all $\mu > 0, \mathscr{A}_\mu \neq \emptyset$. Then, we have the following lemma.

Lemma 3.3. *Under the assumptions (H_1) and (H_2), the following assertions are satisfied.*

(1) *Let $(x_\mu, y_\mu) \in \mathscr{A}_\mu$. If $h(x_\mu, y_\mu) = 0$, then $(x_\mu, y_\mu) \in \mathscr{A}$.*
(2) *If there exists $\mu_0 > 0$, such that $\mathscr{A} \cap \mathscr{A}_{\mu_0} \neq \emptyset$, then for all $\mu > \mu_0$, we have $\mathscr{A} = \mathscr{A}_\mu$.*

Proof

(1) By the definition of the (x_μ, y_μ) and according to the Lemma 3.1, one has $y_\mu \in \mathscr{E}(x_\mu)$, and for all $y \in \mathscr{E}(x) \subset Y(x)$:

$$F(x_\mu, y_\mu) = F(x_\mu, y_\mu) + \mu h(x_\mu, y_\mu) \leqslant F(x, y) + \mu h(x, y).$$

Hence the assertion (1) is satisfied.

(2) Let $(x_{\mu_0}, y_{\mu_0}) \in \mathscr{A} \cap \mathscr{A}_{\mu_0}$, and for all $\mu > \mu_0$, let $(x_\mu, y_\mu) \in \mathscr{A}_\mu$. Then according to the Lemma 3.1 and to the definition of the (x_μ, y_μ) and (x_{μ_0}, y_{μ_0}) one has $F^* = F(x_{\mu_0}, y_{\mu_0}) = F(x_{\mu_0}, y_{\mu_0}) + \mu_0 h(x_{\mu_0}, y_{\mu_0}) \leqslant F(x_\mu, y_\mu) + \mu_0 h(x_\mu, y_\mu) \leqslant F(x_\mu, y_\mu) + \mu h(x_\mu, y_\mu) \leqslant F(x_{\mu_0}, y_{\mu_0}) + \mu h(x_{\mu_0}, y_{\mu_0}) = F^*$.

Therefore, $(\mu - \mu_0)h(x_\mu, y_\mu) = 0$. Since $\mu \neq \mu_0$, it follows that $h(x_\mu, y_\mu) = 0$. Hence $(x_\mu, y_\mu) \in \mathscr{A}$ (assertion 1).

Now, let us show the inverse inclusion. Let $(x^*, y^*) \in \mathscr{A}$, then $x^* \in X^+$ and $y^* \in Y(x^*)$. For $\mu > \mu_0$, Let $(x_\mu, y_\mu) \in \mathscr{A}_\mu$. Then, $(x_\mu, y_\mu) \in \mathscr{A}$. Thus

$$F(x^*, y^*) + \mu h(x^*, y^*) = F(x^*, y^*) = F(x_\mu, y_\mu) = F(x_\mu, y_\mu) + \mu h(x_\mu, y_\mu),$$

hence $x^* \in \mathscr{A}_\mu$. □

Now, we are in a position to show that the penalty is exact.

Theorem 3.3. *Suppose that the assumptions (H_1) and (H_2) are satisfied. Then*

(1) *There exists $\mu^* > 0$, such that the problems (P) and (P_{μ^*}) are equivalent.*
(2) *Assume that for all $x \in V(X^+)$, $Y(x) \neq \mathscr{E}(x)$ (which is equivalent to $V(Y(x)) \setminus \mathscr{E}(x) \neq \emptyset$, where $V(X^+)$ and $V(Y(x))$ denote the sets of the vertices of X^+ and $Y(x)$, respectively). Set*

$$\mu^* = \operatorname{Inf}\{\mu > 0 : \mathscr{A} \cap \mathscr{A}_\mu \neq \emptyset\}, \quad m^* = \underset{\substack{x \in X^+ \\ y \in Y(x)}}{\operatorname{Min}} F(x, y),$$

$$M^* = \underset{\substack{x \in X^+ \\ y \in Y(x)}}{\operatorname{Max}} F(x, y) \quad \text{and} \quad M = \underset{\substack{x \in V(X^+) \\ y \in Y(x) \\ y \notin \mathscr{E}(x) \cap V(Y(x))}}{\operatorname{Min}} h(x, y),$$

then $\frac{M^* - m^*}{M} \geqslant \mu^*$.

Proof

(1) Let $(x_k, y_k) \in V(X^+) \times V(Y(x)$, be a solution of the problem (P_{μ_k}). Let $(x_k, y_k)_{k \in K}$, $K \subset \mathbb{N}$, be a subsequence converging to $(\bar{x}, \bar{y}) \in \mathscr{A}$. Since $V(X^+)$ is finite, it follows that there exists $k_0 \in K$, such that for all $k \geqslant k_0$, $(x_k, y_k) = (\bar{x}, \bar{y})$. Hence, there exists $\mu > 0$, such that $\mathscr{A} \cap \mathscr{A}_\mu \neq \emptyset$. Set

$$\mu^* = \operatorname{Inf}\{\mu > 0 : \mathscr{A} \cap \mathscr{A}_\mu \neq \emptyset\},$$

and let $\mu > \mu^*$. Denote $\mathscr{M} = \{\mu > 0 : \mathscr{A} \cap \mathscr{A}_\mu \neq \emptyset\}$. Then μ is not a lower bound of the set $\mathscr{M}$. Hence, there exists $\mu' < \mu$, such that $\mu' \in \mathscr{M}$. One gets the result by applying the assertion (2) of the Lemma 3.3.

(2) Let $\mu_1 = \frac{M^* - m^*}{M}$, M^* and m^* exist because they are the optimal values of the continuous function F over a polytope. Besides, one has for all $x \in V(X^+)$, $\mathscr{E}(x) \cap V(Y(x)) \neq \emptyset$ (see e.g. Luc, 1989; Theorems 2.10 and 2.11, p. 91). Thus, M exists. According to the Lemma 3.1 and the fact that $M^* - m^* > 0$, one has $M > 0$. Thus, μ_1 exists and $\mu_1 > 0$. Otherwise, let $(x_\mu, y_\mu) \in V(X^+) \times V(x_\mu)$ and $\mu > \mu_1$. By the definition of (x_μ, y_μ), for all $(x, y) \in X^+ \times Y(x)$, one has

$$F(x_\mu, y_\mu) + \mu h(x_\mu, y_\mu) \leqslant F(x, y) + \mu h(x, y).$$

In particular for $(x^*, y^*) \in X^+ \times \mathscr{E}(x^*)$ and by using the assertion (2) of the Lemma 3.1, one also has

$$F(x_\mu, y_\mu) + \mu h(x_\mu, y_\mu) \leqslant F(x^*, y^*) \leqslant M^*. \tag{3.6}$$

Now, suppose that $y_\mu \notin \mathscr{E}(x_\mu)$. Then by the Lemma 3.1, one has $h(x_\mu, y_\mu) > 0$. Hence, for all $\mu > \mu_1$ the inequality (3.6) implies that

$$M^* - m^* \geqslant M^* - F(x_\mu, y_\mu) \geqslant \mu h(x_\mu, y_\mu) > \mu_1 h(x_\mu, y_\mu).$$

By dividing by μ_1, one obtains $M > h(x_\mu, y_\mu)$, which contradicts the definition of M. Hence $y_\mu \in \mathscr{E}(x_\mu)$. The assertion (1) of Lemma 3.1 and the assertion (2) of Lemma 3.3 imply that $(x_\mu, y_\mu) \in \mathscr{A}$. Thus, for all $\mu > \mu_1$, $\mathscr{A} \cap \mathscr{A}_\mu \neq \emptyset$. Hence $\mu_1 \geqslant \mu^*$. □

Now, we present an algorithm model based on the mathematical results above.

Algorithm 3.1.

Initialization:

Define a positive increasing sequence $\mu_k \nearrow +\infty$ and set $k = 0$.

Iteration (k) :

Solve the concave program (P_{μ_k}) and let (x_k, y_k) its solution.

If $h(x_k, y_k) = 0$, then STOP : (x_k, y_k) is a solution of the problem (P).

Else $k \leftarrow k + 1$, and return to iteration k.

Example 3.1. Let

$$F(x, y) = x - 4y, \quad C = (1, 2)^{\mathrm{T}}, \quad X^+ = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\},$$

and

$$Y(x) = \{y \in \mathbb{R}^+ / -x - y \leqslant -3, -2x + 4y \leqslant 0, 2x + y \leqslant 12, -3x + 2y \leqslant -4\}.$$

The problems (P) and $PL(x,y)$ are written as follows:

$$(P) \quad \underset{\substack{x,y \\ x \in X^+ \\ y \in \mathscr{E}(x)}}{\operatorname{Min}} F(x, y) = x - 4y,$$

where $\mathscr{E}(x)$ is the set of the weakly efficient solution of the problem

$$MOP(x) \ \operatorname{Min}_{y \in Y(x)} Cy = (y, 2y)^{\mathrm{T}},$$

$$PL(x, y) \ \operatorname{Max}_{u,t}\{t : (u, t) \in \mathscr{Z}(x, y)\},$$

where $\mathscr{Z}(x, y) = \{(u, t) \in \mathbb{R}^+ \times \mathbb{R} / -u \leqslant x - 3; u \leqslant 2x; u \leqslant -2x + 12; 2u \leqslant 3x - 4; u + t \leqslant y; 2u + t \leqslant 2y\}$. Hence for all $x \in X^+$, $y \in Y(x)$, we have $h(x, y) = 2x + 2y - 6$, and

$$(P_\mu) \begin{cases} \operatorname{Min} \quad (2\mu + 1)x + (2\mu - 4)y - 6\mu \\ \text{s.t} \\ -x - y \leqslant -3, \\ -2x + 4y \leqslant 0, \\ 2x + y \leqslant 12, \\ -3x + 2y \leqslant -4, \\ x \leqslant 3, \\ x, y \geqslant 0. \end{cases}$$

The problem (P_μ) is a parameter linear programming problem. Denote by (x^*_μ, y^*_μ) an optimal solution of the problem (P_μ) and F^*_μ its optimal value. One has the following result

μ	x^*_μ	y^*_μ	F^*_μ	$h(x^*_\mu, y^*_\mu)$
0.1	3	1.5	−2.7	3
0.33	3	1.5	−2.02	3
0.34	2	1	−2	0
1	2	1	−2	0
5	2	1	−2	0
10	2	1	−2	0

Since for all $\mu \geqslant 0.34$, one has $h(x^*_\mu, y^*_\mu) = 0$ then, the point (2, 1) is a solution of the problem (P_μ).

4. All linear case

In the sequel, we consider that the function F is given by

$$F(x,y) = d^{\mathrm{T}}x + g^{\mathrm{T}}y$$

with $d \in \mathbb{R}^n$, $g \in \mathbb{R}^m$.

The problem (BLP) is written as

$$(BLP) \quad \underset{\substack{x,y \\ x\in X, x\geqslant 0 \\ y\in \mathscr{E}(x)}}{\mathrm{Min}} \; d^{\mathrm{T}}x + g^{\mathrm{T}}y,$$

where $\mathscr{E}(x)$ is the set of the weakly efficient solution of the linear multi objective problem $MOP(x)$.

By using the linearity of the function F, we will transform the resolution of the problem (BLP) to the resolution of a sequence of bilinear programming problems.

Now, we give a formulation of the problem (P_μ).

The dual problem of $PL(x, y)$ is

$$DL(x,y)\begin{cases} \mathrm{Min}_{\lambda,\gamma} \quad (b-Ax)^{\mathrm{T}}\gamma + \lambda^{\mathrm{T}}Cy \\ \text{s.t} \\ B^{\mathrm{T}}\gamma + C^{\mathrm{T}}\lambda \geqslant 0, \\ \lambda^{\mathrm{T}}e = 1, \\ \lambda \in \mathbb{R}^r_+, \quad \gamma \in \mathbb{R}^p_+. \end{cases}$$

Since the problem $PL(x, y)$ has a solution (Lemma 2.1), then from the theory of linear programming, $h(x, y)$ is the common optimal value of the problem $PL(x, y)$ and $DL(x, y)$. Then

$$h(x,y) = \mathrm{Min}_{(\lambda,\gamma)\in\mathscr{D}} (b-Ax)^{\mathrm{T}}\gamma + \lambda^{\mathrm{T}}Cy,$$

where $\mathscr{D} = \{(\lambda,\gamma) \in \mathbb{R}^r_+ \times \mathbb{R}^m_+ / \lambda^{\mathrm{T}}e = 1, C^{\mathrm{T}}\lambda + B^{\mathrm{T}}\gamma \geqslant 0\}$.

Replacing $h(x, y)$ by its expression, the problem (P_μ) becomes

$$\underset{\substack{x,y \\ x\in X^+ \\ y\in Y(x)}}{\mathrm{Min}} (d^{\mathrm{T}}x + g^{\mathrm{T}}y) + \underset{(\lambda,\gamma)\in\mathscr{D}}{\mathrm{Min}} \mu(b-Ax)^{\mathrm{T}}\gamma + \mu\lambda^{\mathrm{T}}Cy,$$

that is also equivalent to

$$\begin{cases} \mathrm{Min}_{x,y,\lambda,\gamma} \quad d^{\mathrm{T}}x + g^{\mathrm{T}}y + \mu(b-Ax)^{\mathrm{T}}\gamma + \mu\lambda^{\mathrm{T}}Cy \\ \text{s.t} \\ x \in X^+, \\ y \in Y(x), \\ (\lambda,\gamma) \in \mathscr{D}. \end{cases}$$

Then, if we use the explicit form of $\mathscr{D}$ and $Y(x)$, we obtain the following parameter bilinear program

$$(P_\mu)\begin{cases} \mathrm{Min}_{x,y,\lambda,\gamma} \quad d^{\mathrm{T}}x + g^{\mathrm{T}}y + \mu(b-Ax)^{\mathrm{T}}\gamma + \mu\lambda^{\mathrm{T}}Cy \\ \text{s.t} \\ Ax + By \leqslant b, \\ C^{\mathrm{T}}\lambda + B^{\mathrm{T}}\gamma \geqslant 0, \\ \lambda^{\mathrm{T}}e = 1, \\ \lambda \in \mathbb{R}^r_+, \quad \gamma \in \mathbb{R}^p_+, \\ x \in X^+, \quad y \in \mathbb{R}^m_+. \end{cases}$$

Now, we present an algorithm, in which one solve at each iteration just a bilinear program without computing the function h.

Algorithm 4.1

Initialization :

Define a positive increasing sequence $\mu_k \nearrow +\infty$ and set $k = 0$.

Iteration (k):

Solve the bilinear program (P_{μ_k}) and let $(x_k, y_k, \lambda_k, \gamma_k)$ its solution.

If

$$(b-Ax)^{\mathrm{T}}\gamma + \lambda^{\mathrm{T}}Cy = 0, \qquad (\mathrm{C})$$

then STOP : (x_k, y_k) is a solution of the problem (P).

Else $k \leftarrow k + 1$, and return to iteration k.

Remark 4.1. The condition $(\mathscr{C})$ is merely the condition $h(x_k, y_k) = 0$ (assertion 1) of the Lemma 3.3).

Remark 4.2. Many authors propose algorithms that solve the bilinear problem (P_{μ_k}). The reader can refer to Gallo and Ulkucco (1977), Konno (1976).

Example 4.1. We come back to the problem from the Example 4.1. One has $d = 1$, $g = -4$, $A = (-1, -2, 2, -3)^{\mathrm{T}}$, $B = (-1, 4, 1, 2)^{\mathrm{T}}$, $C = (1, 2)^{\mathrm{T}}$, $b = (-3, 0, 12, -4)^{\mathrm{T}}$.

Then $(b - Ax)^{\mathrm{T}} = (-3 + x, \ 2x, \ 12 - 2x, \ -4 + 3x)$, and $\lambda^{\mathrm{T}}Cy = \lambda_1 y + 2\lambda_2 y$. Denote by f_μ the objective function of the problem (P_μ), thus

$$f_\mu(x,y,\lambda,\gamma) = x - 4y - 3\mu\gamma_1 + 12\mu\gamma_3 - 4\mu\gamma_4 + \mu x(\gamma_1 + 2\gamma_2 - 2\gamma_3 + 3\gamma_4) + \mu y(\lambda_1 + 2\lambda_2).$$

Hence, the problem (P_μ) can be written as

$$(P_\mu)\begin{cases} \mathrm{Min}_{x,y\lambda,\gamma} \quad f_\mu(x,y,\lambda,\gamma) \\ \text{s.t} \\ \lambda_1 + 2\lambda_2 - \gamma_1 + 4\gamma_2 + \gamma_3 + 2\gamma_4 \geqslant 0, \\ -x - y \leqslant -3, \\ -2x + 4y \leqslant 0, \\ 2x + y \leqslant 12, \\ -3x + 2y \leqslant -4, \\ x \leqslant 3, \\ \lambda_1 + \lambda_2 = 1, \\ \lambda \in \mathbb{R}^2_+, \quad \gamma \in \mathbb{R}^4_+, \\ x \in \mathbb{R}_+, \quad y \in \mathbb{R}_+. \end{cases}$$

Let $(x^*_\mu, y^*_\mu, \lambda^*_\mu, \gamma^*_\mu)$ be an optimal solution of the problem (P_μ), F^*_μ its optimal value and $a = (b - Ax^*_\mu)^{\mathrm{T}}\gamma^*_\mu + \lambda^{\mathrm{T}}Cy^*_\mu$. We have the following result

μ	x^*_μ	y^*_μ	$\gamma^*_{\mu1}$	$\gamma^*_{\mu2}$	$\gamma^*_{\mu3}$	$\gamma^*_{\mu4}$	$\lambda^*_{\mu1}$	$\lambda^*_{\mu2}$	F^*_μ	a
0.1	3	1.5	0	0	0	0	1	0	–2.85	1.5
0.4	3	1.5	0	0	0	0	1	0	–2.4	1.5
0.66	3	1.5	0	0	0	0	1	0	–2.01	1.5
0.67	2	1	1	0	0	0	1	0	–2	0
1	2	1	1	0	0	0	1	0	–2	0
5	2	1	1	0	0	0	1	0	–2	0
100	2	1	1	0	0	0	1	0	–2	0

For all $\mu \geqslant 0.67$, one has $(b - Ax^*_\mu)^{\mathrm{T}}\gamma^*_\mu + \lambda^{\mathrm{T}}Cy^*_\mu = 0$ then, the point (2.1) is a solution of the problem (P).

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