# Matrix Completion With Deterministic Pattern: A Geometric Perspective

Alexander Shapiro, Yao Xie<sup>10</sup>, and Rui Zhang

Abstract—We consider the matrix completion problem with a deterministic pattern of observed entries. In this setting, we aim to answer the question: Under what conditions will there be (at least locally) unique solution to the matrix completion problem, i.e., the underlying true matrix is identifiable? We answer the question from a certain point of view and outline a geometric perspective. We give an algebraically verifiable sufficient condition, which we call the well-posedness condition, for the local uniqueness of minimum rank matrix completion (MRMC) solutions. It appears that this condition is basic for the analysis of MRMC, and we show that, in a sense, the condition is generic. We also argue that the low-rank approximation approaches are more stable than MRMC and further propose a sequential statistical testing procedure to determine the "true" rank from observed entries. Finally, we provide numerical examples aimed at verifying validity of the presented theory.

*Index Terms*—Linear algebra, matrices, estimation theory, mathematical programming, least squares approximation.

### I. INTRODUCTION

ATRIX completion (e.g., [1]–[3]) is a fundamental problem in signal processing and machine learning, which studies the recovery of a low-rank matrix from an observation of a subset of its entries. It has attracted a lot attention from researchers and practitioners and there are various motivating real-world applications including recommender systems and the Netflix challenge (see a recent overview in [4]). A popular approach for matrix completion is to find a matrix of minimal rank satisfying the observation constraints. Due to the non-convexity of the rank function, popular approaches are convex relaxation (see, e.g., [5]) and nuclear norm minimization. There is a rich literature, both in establishing performance bounds, developing efficient algorithms and providing performance guarantees. Recently there has also been new various results for non-convex formulations of matrix completion problem (see, e.g., [6]).

The authors are with the H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332 USA (e-mail: ashapiro@isye.gatech.edu; yao.xie@isye.gatech.edu; ruizhang\_ray@ gatech.edu).

Digital Object Identifier 10.1109/TSP.2018.2885494

Existing conditions ensuring recovery of the minimal rank matrix are usually formulated in terms of missing-at-random entries and under an assumption of the so-called bounded-coherence (see a survey for other approaches in [4]; we do not aim to give a complete overview of the vast literature). These results are typically aimed at establishing the recovery with a high probability. In addition, there has been much work on low-rank matrix recovery (see, e.g., [7], which studies a related problem: the uniqueness conditions for minimum rank matrix; here the linear measurements of the true matrix; here the linear measurements correspond to inner product of a measurement mask matrix with the true matrix, and hence, the observations are different from that in matrix completion).

With a deterministic pattern of observed entries, a complete characterization of the identifiable matrix for matrix completion remains an important yet open question: under what conditions for the pattern, there will be (at least locally) unique solution? Recent work [8] provides insights into this problem by studying the so-called completable problems and establishing conditions ensuring the existence of at most finitely many rank-*r* matrices that agree with all its observed entries. A related work [9] studied this problem when there is a sparse noise that corrupts the entries. The rank estimation problem has been discussed in [10], [11], and related tensor completion problem in [12]: the goal in these works are different though; they aim to find upper and lower bounds for the true rank, whereas our rank selection test in Section IV determines the most plausible rank from a statistical point of view.

In this paper, we aim to answer the question from a somewhat different point of view and to give a geometric perspective. In particular, we consider the solution of the Minimum Rank Matrix Completion (MRMC) formulation, which leads to a non-convex optimization problem. We address the following questions: (i) Given observed entries arranged according to a (deterministic) pattern, by solving the MRMC problem, what is the minimum achievable rank? (ii) Under what conditions, there will be a unique matrix that is a solution to the MRMC problem? We give a sufficient condition (which we call the *well-posedness condition*) for the local uniqueness of MRMC solutions, and illustrate how such condition can be verified. We also show that such well-posedness condition in a sense is generic. In addition, we also consider the convex relaxation and nuclear norm minimization formulations.

It is well known that matrices of rank r form a smooth manifold of dimension  $r(n_1 + n_2 - r)$  in the linear space of  $n_1 \times n_2$  matrices (e.g., [13, Example 2.5.10]). Consequently

1053-587X © 2018 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications\_standards/publications/rights/index.html for more information.

Manuscript received February 21, 2018; revised August 27, 2018 and October 22, 2018; accepted November 16, 2018. Date of publication December 10, 2018; date of current version January 9, 2019. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Yuantao Gu. The work of A. Shapiro was supported by the National Science Foundation under Grant 1633196 and DARPA EQUIPS program, and under Grant SNL 014150709. The work of Y. Xie was supported in part by the National Science Foundation under Grants CCF-1442635, CMMI-1538746, DMS-1830210, in part by the National Science Foundation CAREER Award CCF-1650913, and in part by the S.F. Express Award. (*Corresponding author: Yao Xie.*)

we argue that given *m* observations of an  $n_1 \times n_2$  matrix, if the minimal rank  $r^*$  is less than  $\Re(n_1, n_2, m) := (n_1 + n_2)/2 - [(n_1 + n_2)^2/4 - m]^{1/2}$ , then the corresponding solution is unstable in the sense that an arbitrary small perturbation of the observed values can make this rank unattainable. On the other hand if  $r^* > \Re(n_1, n_2, m)$ , then almost surely the solution is not (even locally) unique (cf., [14]). This indicates that except on rare occasions, the MRMC problem cannot have both properties of possessing unique and stable solutions. Consequently, what makes sense is to try to solve the minimum rank problem approximately and hence to consider low-rank approximation approaches (such as an approach mentioned in [4], [15]) as a better alternative to the MRMC formulation.

We also propose a sequential statistical testing procedure to determine the 'true' rank from noisy observed entries. Such statistical approach can be useful for many existing low-rank matrix completion algorithms, which require a pre-specification of the matrix rank, such as the alternating minimization approach to solving the non-convex problem by representing the low-rank matrix as a product of two low-rank matrix factors (see, e.g., [4], [16], [17]).

The paper is organized as follows. In the next section, we introduce the considered setting and some basic definitions. In Section II we present the problem set-up, including the MRMC, LRMA, and convex relaxation formulations. Section III contains the main theoretical results. A statistical test of rank is presented in Section IV. In Section V we present numerical results related to the developed theory. Finally Section VI concludes the paper. All proofs are transferred to the Appendix.

We use conventional notations. For  $a \in \mathbb{R}$  we denote by  $\lceil a \rceil$  the least integer that is greater than or equal to a. By  $A \otimes B$  we denote the Kronecker product of matrices (vectors) A and B, and by vec(A) column vector obtained by stacking columns of matrix A. We use the following matrix identity for matrices A, B, C of appropriate order

$$\operatorname{vec}(ABC) = (C^{\top} \otimes A)\operatorname{vec}(B).$$
 (1)

By  $\mathbb{S}^p$  we denote the linear space of  $p \times p$  symmetric matrices and by writing  $X \succeq 0$  we mean that matrix  $X \in \mathbb{S}^p$  is positive semidefinite. By  $\sigma_i(Y)$  we denote the *i*-th largest singular value of matrix  $Y \in \mathbb{R}^{n_1 \times n_2}$ . By  $I_p$  we denote the identity matrix of dimension p.

#### II. MATRIX COMPLETION AND PROBLEM SET-UP

Consider the problem of recovering an  $n_1 \times n_2$  data matrix of low rank when observing a small number m of its entries, which are denoted as  $M_{ij}$ ,  $(i, j) \in \Omega$ . We assume that  $n_1 \ge 2$ and  $n_2 \ge 2$ . Here  $\Omega \subset \{1, ..., n_1\} \times \{1, ..., n_2\}$  is an index set of cardinality m. The low-rank matrix completion problem, or matrix completion problem, aims to infer the missing entries, based on the available observations  $M_{ij}$ ,  $(i, j) \in \Omega$ , by using a matrix whose rank is as small as possible.

Low-rank matrix completion problem is usually studied under a missing-at-random model, under which the necessary and sufficient conditions for perfect recovery of the true matrix are known [18]–[23]. Study of deterministic sampling pattern is relatively rare. This includes the finitely rank-*r* completability problem in [8], which shows the conditions for the deterministic sampling pattern such that there exists at most *finitely many* rank-r matrices that agrees with its observed entries. In this paper, we study a related but different problem, i.e., when will the matrix have a unique way to be completed, given a fixed sampling pattern. This is a fundamental problem related to *the identifiability* of a low-rank matrix given an observation pattern  $\Omega$ .

#### A. Definitions

Let us introduce some necessary definitions. Denote by Mthe  $n_1 \times n_2$  matrix with the specified entries  $M_{ij}$ ,  $(i, j) \in \Omega$ , and all other entries equal zero. Consider  $\Omega^c := \{1, ..., n_1\} \times \{1, ..., n_2\} \setminus \Omega$ , the complement of the index set  $\Omega$ , and define

$$\mathbb{V}_{\Omega} := \left\{ Y \in \mathbb{R}^{n_1 \times n_2} : Y_{ij} = 0, \ (i,j) \in \Omega^c \right\}$$

This linear space represents the set of matrices that are filled with zeros at the locations of the unobserved entries. Similarly define

$$\mathbb{V}_{\Omega^c} := \left\{ Y \in \mathbb{R}^{n_1 imes n_2} : Y_{ij} = 0, \; (i,j) \in \Omega 
ight\}.$$

By  $P_{\Omega}$  we denote the projection onto the space  $\mathbb{V}_{\Omega}$ , i.e.,  $[P_{\Omega}(Y)]_{ij} = Y_{ij}$  for  $(i, j) \in \Omega$  and  $[P_{\Omega}(Y)]_{ij} = 0$  for  $(i, j) \in \Omega^c$ . By this construction,  $\{M + X : X \in \mathbb{V}_{\Omega^c}\}$  is the affine space of all matrices that satisfy the observation constraints. Note that  $M \in \mathbb{V}_{\Omega}$  and the dimension of the linear space  $\mathbb{V}_{\Omega}$  is  $\dim(\mathbb{V}_{\Omega}) = m$ , while  $\dim(\mathbb{V}_{\Omega^c}) = n_1 n_2 - m$ .

We say that a property holds for *almost every* (a.e.)  $M_{ij}$ , or almost surely, if the set of matrices  $Y \in \mathbb{V}_{\Omega}$  for which this property does not hold has Lebesgue measure zero in the space  $\mathbb{V}_{\Omega}$ .

#### B. Minimum Rank Matrix Completion (MRMC)

Since the true rank is unknown, a natural approach is to find the minimum rank matrix that is consistent with the observations. This goal can be written as the following optimization problem referred to as the Minimum Rank Matrix Completion (MRMC),

$$\min_{Y \in \mathbb{R}^{n_1 \times n_2}} \operatorname{rank}(Y) \text{ subject to } Y_{ij} = M_{ij}, \ (i,j) \in \Omega.$$
 (2)

In general, the rank minimization problem is non-convex and NP-hard to solve. However, this problem is fundamental to various efficient heuristics derived from here. Largely, there are two categories of approximation heuristics: (i) approximate the rank function with some surrogate function such as the nuclear norm function, (ii) or solve a sequence of rank-constrained problems such as the matrix factorization based method, which we will discuss below. Approach (ii) requires to specify the target rank of the recovered matrix beforehand, which we will present a novel statistical test next.

## C. Low Rank Matrix Approximation (LRMA)

Consider the problem

¥

$$\min_{Y \in \mathbb{R}^{n_1 \times n_2}, X \in \mathbb{V}_{\Omega^c}} F(M + X, Y) \text{ s.t. } \operatorname{rank}(Y) = r, \quad (3)$$

where  $M \in \mathbb{V}_{\Omega}$  is the given data matrix, and F(A, B) is a discrepancy between matrices  $A, B \in \mathbb{R}^{n_1 \times n_2}$ . For example, let  $F(A, B) := ||A - B||_F^2$  with  $||Y||_F^2 = \operatorname{tr}(Y^\top Y) = \sum_{i,j} Y_{ij}^2$ , being the Frobenius norm. Define the set of  $n_1 \times n_2$  matrices of rank r

$$\mathcal{M}_r := \left\{ Y \in \mathbb{R}^{n_1 \times n_2} : \operatorname{rank}(Y) = r \right\}$$
(4)

Then (3) becomes the least squares problem

$$\min_{Y \in \mathcal{M}_r} \sum_{(i,j) \in \Omega} \left( M_{ij} - Y_{ij} \right)^2.$$
(5)

The least squares approach although is natural, is not the only one possible. For example, in the statistical approach to Factor Analysis the discrepancy function is based on the Maximum Likelihood method and is more involved (e.g., [24]).

#### D. SDP Formulations: Trace and Nuclear Norm Minimization

An alternative approach to the MRMC problem, which has been studied extensively in the literature, is the convex relaxation formulation (e.g., [1], [5]). Let  $\mathfrak{S} \subset \{1, ..., p\} \times \{1, ..., p\}$  be the symmetric index set corresponding to the index set  $\Omega$ , i.e.,  $(i, n_1 + j) \in \mathfrak{S}$  when  $1 \le i \le n_1$ , if and only if  $(i, j) \in \Omega$ ; and if  $(i, j) \in \mathfrak{S}$ , then  $(j, i) \in \mathfrak{S}$ . By  $\mathfrak{S}^c \subset \{1, ..., p\} \times \{1, ..., p\}$ we denote the symmetric index set complement of  $\mathfrak{S}$ . Define

$$\mathbb{W}_{\mathfrak{S}} := \{ X \in \mathbb{S}^p : X_{ij} = 0, \ (i,j) \in \mathfrak{S}^c \}$$

and

$$\mathbb{W}_{\mathfrak{S}^c} := \{ X \in \mathbb{S}^p : X_{ij} = 0, \ (i,j) \in \mathfrak{S} \}.$$

Define  $\Xi \in \mathbb{S}^p$ ,  $p = n_1 + n_2$ , a symmetric matrix of the following form, that contains the data,

$$\Xi = \begin{bmatrix} 0 & M \\ M^\top & 0 \end{bmatrix}$$

The MRMC problem (2) can be formulated in the following equivalent form

$$\min_{X \in \mathbb{W}_{\mathfrak{S}^c}} \operatorname{rank}(\Xi + X) \text{ subject to } \Xi + X \succeq 0.$$
 (6)

Minimization in (6) is performed over matrices  $X \in \mathbb{S}^p$  which are complement to  $\Xi$  in the sense of having zero entries at all places corresponding to the specified values  $M_{ij}$ ,  $(i, j) \in \Omega$ . We consider a more general minimum rank problem of the form (6) in that we allow the index set  $\mathfrak{S}$  to be a general symmetric subset of  $\{1, ..., p\} \times \{1, ..., p\}$ , with a given matrix  $\Xi \in \mathbb{W}_{\mathfrak{S}}$ . Note that  $\mathbb{W}_{\mathfrak{S}} \cap \mathbb{W}_{\mathfrak{S}^c} = \{0\}$  and  $\mathbb{W}_{\mathfrak{S}} + \mathbb{W}_{\mathfrak{S}^c} = \mathbb{S}^p$ .

As a heuristic it was suggested in [5] to approximate problem (6) by the following trace minimization problem

$$\min_{X \in \mathbb{W}_{\mathfrak{S}^c}} \operatorname{tr}(X) \text{ subject to } \Xi + X \succeq 0, \tag{7}$$

which is equivalent to the following *nuclear norm minimization* problem

$$\min_{Y} \|X + M\|_* \text{ subject to } X \in \mathbb{V}_{\Omega^c}.$$
(8)

Problem (7) is a special case of the following general SDP problem (if we introduce a weight matrix  $C \in W_{\mathfrak{S}^c}$ ):

$$\min_{X \in \mathbb{W}_{\mathfrak{S}^c}} \operatorname{tr}(CX) \text{ subject to } \Xi + X \succeq 0.$$
 (9)

The above formulation is a semidefinite programming (SDP) problem and can be solved efficiently, e.g., by using the singular value thresholding algorithm [25]. Therefore, it has been commonly adopted as an approximation to the minimum rank problem.

*Remark II.1:* Positive semidefinite matrices of rank r form a face of  $\mathbb{S}^p$  of dimension r(r+1)/2. It follows that problem (6) has an optimal solution of rank r such that  $r(r+1)/2 + \dim(\mathbb{W}_{\mathfrak{S}^c}) \leq p(p+1)/2$  (cf., [26]). Since  $p = n_1 + n_2$  and  $\dim(\mathbb{W}_{\mathfrak{S}^c}) = n_1(n_1+1)/2 + n_2(n_2+1)/2 + n_1n_2 - m$ , it follows that problem (7), and hence the nuclear norm minimization problem (8), has an optimal solution of rank r such that  $r(r+1)/2 \leq m$ .

## **III. MAIN THEORETICAL RESULTS**

To gain insights into the identifiability issue of matrix completion, we aim to answer the following two related questions: (i) what is achievable minimum rank (the optimal value of problem (2)), and (ii) whether the minimum rank matrix, i.e., the optimal solutions to (2), is unique given a problem set-up. These result will also help to gain insights in the tradeoff in the theoretical properties of other matrix completion formulations, including LRMA and SDP formulations, compared with the original MRMC formulation.

We show that given  $m = |\Omega|$  observations of an  $n_1 \times n_2$  matrix: (i) if the minimal rank  $r^*$  is less than  $\Re(n_1, n_2, m) := (n_1 + n_2)/2 - [(n_1 + n_2)^2/4 - m]^{1/2}$ , then the corresponding solution is unstable: an arbitrary small perturbation of the observed values can make this rank unattainable; (ii) if  $r^* > \Re(n_1, n_2, m)$ , then almost surely the solution is not (even locally) unique (cf., [14]). This indicates that except in rare occasions, the MRMC problem cannot have both properties of possessing unique and stable solutions. Consequently, LRMA approaches (also used in [4], [15]) could be a better alternative to the MRMC formulation. Moreover, we argue that the nuclear norm minimization approach is not statistically efficient (Section III-H).

#### A. Rank Reducibility

We denote by  $r^*$  the optimal value of problem (2). That is,  $r^*$  is the minimal rank of an  $n_1 \times n_2$  matrix with prescribed elements  $M_{ij}$ ,  $(i, j) \in \Omega$ . Clearly,  $r^*$  depends on the index set  $\Omega$  and values  $M_{ij}$ . A natural question is what values of  $r^*$  can be attained. Recall that (2) is a non-convex problem and may have multiple solutions.

In a certain generic sense it is possible to give a lower bound for the minimal rank  $r^*$ . Let us consider intersection of a set of low-rank matrices and the affine space of matrices satisfying the observation constraints. Define the (affine) mapping  $\mathcal{A}_M$ :  $\mathbb{V}_{\Omega^c} \to \mathbb{R}^{n_1 \times n_2}$  as

$$\mathcal{A}_M(X) := M + X, \ X \in \mathbb{V}_{\Omega^c}.$$

As it has been pointed out before, the image  $\mathcal{A}_M(\mathbb{V}_{\Omega^c}) = M + \mathbb{V}_{\Omega^c}$  of mapping  $\mathcal{A}_M$  defines the space of feasible points of the MRMC problem (2). It is well known that  $\mathcal{M}_r$  is a smooth,  $C^{\infty}$ ,

manifold with

$$\dim(\mathcal{M}_r) = r(n_1 + n_2 - r).$$
(10)

It is said that the mapping  $\mathcal{A}_M$  intersects  $\mathcal{M}_r$  transverally if for every  $X \in \mathbb{V}_{\Omega^c}$  either  $\mathcal{A}_M(X) \notin \mathcal{M}_r$ , or  $\mathcal{A}_M(X) \in \mathcal{M}_r$  and the following condition holds

$$\mathbb{V}_{\Omega^c} + \mathcal{T}_{\mathcal{M}_r}(Y) = \mathbb{R}^{n_1 \times n_2}, \qquad (11)$$

where  $Y := \mathcal{A}_M(X)$  and  $\mathcal{T}_{\mathcal{M}_r}(Y)$  denotes the tangent space to  $\mathcal{M}_r$  at  $Y \in \mathcal{M}_r$  (we will give explicit formulas for the tangent space  $\mathcal{T}_{\mathcal{M}_r}(Y)$  in equations (18) and (19) below).

By using a classical result of differential geometry, it is possible to show that for *almost every* (a.e.)  $M_{ij}$ ,  $(i, j) \in \Omega$ , the mapping  $\mathcal{A}_M$  intersects  $\mathcal{M}_r$  transverally (this holds for every r) (see [14] for a discussion of this result). Transversality condition (11) means that the linear spaces  $\mathbb{V}_{\Omega^c}$  and  $\mathcal{T}_{\mathcal{M}_r}(Y)$ together span the whole space  $\mathbb{R}^{n_1 \times n_2}$ . Of course this cannot happen if the sum of their dimensions is less than the dimension of  $\mathbb{R}^{n_1 \times n_2}$ . Therefore transversality condition (11) implies the following dimensionality condition

$$\dim(\mathbb{V}_{\Omega^c}) + \dim(\mathcal{T}_{\mathcal{M}_r}(Y)) \ge \dim(\mathbb{R}^{n_1 \times n_2}).$$
(12)

In turn the above condition (12) can be written as

$$r(n_1 + n_2 - r) \ge m,$$
 (13)

or equivalently  $r \geq \Re(n_1, n_2, mm)$ , where

$$\Re(n_1, n_2, m) := (n_1 + n_2)/2 - \sqrt{(n_1 + n_2)^2/4 - m}.$$
 (14)

That is, if  $r < \Re(n_1, n_2, m)$ , then the transversality condition (11) cannot hold and hence for a.e.  $M_{ij}$  it follows that  $\operatorname{rank}(M + X) \neq r$  for all  $X \in \mathbb{V}_{\Omega^c}$ .

Now if  $\mathcal{A}_M$  intersects  $\mathcal{M}_r$  transverally at  $\mathcal{A}_M(X) \in \mathcal{M}_r$ (i.e., condition (11) holds), then the intersection  $\mathcal{A}_M(\mathbb{V}_{\Omega^c}) \cap \mathcal{M}_r$  forms a smooth manifold near the point  $Y := \mathcal{A}_M(X)$ . When  $r > \mathfrak{R}(n_1, n_2, m)$ , this manifold has dimension greater than zero and hence the corresponding rank r solution is not (locally) unique. This leads to the following (for a formal discussion of these results we can refer to [14]).

Theorem III.1 (Generic lower bound and non-uniqueness of solutions): For any index set  $\Omega$  of cardinality m and almost every  $M_{ij}$ ,  $(i, j) \in \Omega$ , the following holds: (i) for every feasible point Y of problem (2) it follows that

$$\operatorname{rank}(Y) \ge \Re(n_1, n_2, m), \tag{15}$$

(ii) if  $r^* > \Re(n_1, n_2, m)$ , then problem (2) has multiple (more than one) optimal solutions.

It follows from part (i) of Theorem III.1 that  $r^* \ge \Re(n_1, n_2, m)$  for a.e.  $M_{ij}$ . Generically (i.e., almost surely) the following lower bound for the minimal rank  $r^*$  holds

$$r^* \ge \Re(n_1, n_2, m),\tag{16}$$

and (2) may have unique optimal solution only when  $r^* = \Re(n_1, n_2, m)$ . Of course such equality could happen only if  $\Re(n_1, n_2, m)$  is an integer number. As Example III.1 below shows, for any integer  $r^* \leq \lceil \sqrt{m} \rceil$  satisfying (16), there exists an index set  $\Omega$  such that the corresponding MRMC problem attains the minimal rank  $r^*$  for a.e.  $M_{ij}$ . In particular this shows

that the lower bound (16) is tight. When we have a square matrix  $n_1 = n_2 = n$ , it follows that

$$\Re(n,n,m) = n - \sqrt{n^2 - m}.$$
(17)

For  $n_1 = n_2 = n$  and small  $m/n^2$  we can approximate

$$\Re(n,n,m) = n\left(1 - \sqrt{1 - m/n^2}\right) \approx m/(2n).$$

For example, for  $n_1 = n_2 = 1000$  and m = 20000 we have  $\Re(n, n, m) = 10.05$ , and hence the bound (16) becomes  $r^* \ge 11$ . The nuclear norm minimization guarantees to recover a solution of rank  $r \le 199$  (see Remark II.1).

*Example III.1 (Tightness of the lower bound for*  $r^*$ ): For r $< \min\{n_1, n_2\}$  consider data matrix M of the following form  $M = \begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix}$ . Here, the three sub-matrices  $M_1, M_2, M_3$ , of the respective order  $r \times r$ ,  $(n_1 - r) \times r$  and  $(n_1 - r) \times r$  $(n_2 - r)$ , represent the observed entry values. Cardinality m of the corresponding index set  $\Omega$  is  $r(n_1 + n_2 - r)$ , i.e., here  $r = \Re(n_1, n_2, m)$ . Suppose that the  $r \times r$  matrix  $M_1$  is nonsingular, i.e., its rows are linearly independent. Then any row of matrix  $M_2$  can be represented as a (unique) linear combination of rows of matrix  $M_1$ . It follows that the corresponding MRMC problem has (unique) solution of rank  $r^* = r$ . In other words, the rank of the completed matrix will be equal to r (the rank of the sub-matrix  $M_1$ ) and there will be a unique matrix that achieves this rank. Now suppose that some of the entries of the matrices  $M_2$  and  $M_3$  are not observed, and hence cardinality of the respective index set  $\Omega$  is less than  $r(n_1 + n_2 - r)$ , and thus  $r > \Re(n_1, n_2, m)$ . In that case the respective minimal rank still is r, provided matrix  $M_1$  is nonsingular, although the corresponding optimal solutions are not unique. In particular, if  $M = \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix}$ , i.e., only the entries of matrix  $M_1$  are observed, then  $m = r^2$  and the minimum rank is r.

## B. Uniqueness of Solutions of the MRMC Problem

Following Theorem III.1, for a given matrix  $M \in \mathbb{V}_{\Omega}$  and the corresponding minimal rank  $r^* \leq \mathfrak{R}(n_1, n_2, m)$ , the question is whether the corresponding solution  $Y^*$  of rank  $r^*$  is unique. Although, the set of such matrices M is "thin" (in the sense that it has Lebesgue measure zero), this question of uniqueness is important, in particular for the statistical inference of rank (discussed in Section IV). Available results, based on the so-called Restricted Isometry Property (RIP) for low-rank matrix recovery from linear observations and based on the coherence property for low-rank matrix completion, assert that for certain probabilistic (Gaussian) models such uniqueness holds with high probability. However for a given matrix  $M \in \mathbb{V}_{\Omega}$  it could be difficult to verify whether the solution is unique (some sufficient conditions for such uniqueness are given in [8, Theorem 2], we will comment on this below.)

Let us consider the following concept of local uniqueness of solutions.

Definition III.1: We say that an  $n_1 \times n_2$  matrix  $\bar{Y}$  is a locally unique solution of problem (2) if  $P_{\Omega}(\bar{Y}) = M$  and there is a neighborhood  $\mathcal{V} \subset \mathbb{R}^{n_1 \times n_2}$  of  $\bar{Y}$  such that  $\operatorname{rank}(Y) \neq \operatorname{rank}(\bar{Y})$ for any  $Y \in \mathcal{V}$ ,  $P_{\Omega}(Y) = M$  and  $Y \neq \bar{Y}$ . 1092



Fig. 1. Illustration of the well-posedness condition.

Note that rank is a lower semicontinuous function of matrix, i.e., if  $\{Y_k\}$  is a sequence of matrices converging to matrix Y, then  $\liminf_{k\to\infty} \operatorname{rank}(Y_k) \ge \operatorname{rank}(Y)$ . Therefore local uniqueness of  $\overline{Y}$  actually implies existence of the neighborhood  $\mathcal{V}$  such that  $\operatorname{rank}(Y) > \operatorname{rank}(\overline{Y})$  for all  $Y \in \mathcal{V}, Y \neq \overline{Y}$ , i.e., that at least locally problem (2) does not have optimal solutions different from  $\overline{Y}$ . The definition (III.1) is closely related to the *finitely rank-r completability* condition introduced in [8], which assumes that the MRMC problem has a finite number of rank r solutions. Of course if problem (2) has a non locally unique solution of rank r, then the finitely rank-r completability condition cannot hold.

We now introduce some constructions associated with the manifold  $\mathcal{M}_r$  of matrices of rank r. There are several equivalent forms how the tangent space to the manifold  $\mathcal{M}_r$  at  $Y \in \mathcal{M}_r$  can be represented. Let  $Y = VW^{\top}$  for some matrices  $V \in \mathbb{R}^{n_1 \times r}$  and  $W \in \mathbb{R}^{n_2 \times r}$  of rank r. Then

$$\mathcal{T}_{\mathcal{M}_r}(Y) = \left\{ (dV)W^\top + V(dW)^\top : \\ dV \in \mathbb{R}^{n_1 \times r}, \ dW \in \mathbb{R}^{n_2 \times r} \right\}.$$
(18)

In an equivalent form this tangent space can be written as

$$\mathcal{T}_{\mathcal{M}_r}(Y) = \left\{ H \in \mathbb{R}^{n_1 \times n_2} : FHG = 0 \right\},\tag{19}$$

where F is an  $(n_1 - r) \times n_1$  matrix of rank  $n_1 - r$  such that FY = 0 (referred to as a *left side complement* of Y) and G is an  $n_2 \times (n_2 - r)$  matrix of rank  $n_2 - r$  such that YG = 0 (referred to as a *right side complement* of Y). We also use the linear space of matrices orthogonal (normal) to  $\mathcal{M}_r$  at  $Y \in \mathcal{M}_r$ , denoted by  $\mathcal{N}_{\mathcal{M}_r}(Y)$ . By (18) it follows that

$$\mathcal{N}_{\mathcal{M}_r}(Y) = \left\{ Z \in \mathbb{R}^{n_1 \times n_2} : Z^\top Y = 0 \text{ and } Y Z^\top = 0 \right\}.$$
(20)

Definition III.2 (Well-posedness condition): We say that a matrix  $\bar{Y} \in \mathcal{M}_r$  is well-posed, for problem (2), if  $P_{\Omega}(\bar{Y}) = M$  and the following condition holds

$$\mathbb{V}_{\Omega^c} \cap \mathcal{T}_{\mathcal{M}_r}(\bar{Y}) = \{0\}.$$
(21)

Condition (21) (illustrated in Figure 1) is a natural condition having a simple geometrical interpretation. Intuitively, it means that the null space of the observation operator does not have any non-trivial matrix that lies in the tangent space of low-rank matrix manifold. Hence, there cannot be any local deviations from the optimal solution that still satisfy the measurement constraints. This motivates us to introduce the well-posedness condition that guarantees a matrix to be locally unique solution. Note that this is different from the so-called Null Space Property (NSP) (see, e.g., [27]). Although both the well-posedness and NSP have a similar geometrical flavor, the NSP is aimed at ensuring uniqueness of solution of a convex problem, while the MRMC is essentially a nonconvex construction. For instance, NSP can be used to guarantee uniqueness of solutions of the optimization problem  $\min_{x \in \mathbb{R}^n} ||x||_1$  subject to Ax = b, which can be formulated as a linear programming problem.

Now we can give sufficient conditions for local uniqueness:

Theorem III.2 (Sufficient conditions for local uniqueness): Matrix  $\overline{Y} \in \mathcal{M}_r$  is a locally unique solution of problem (2) if  $\overline{Y}$  is well-posed for (2).

#### C. Verifiable form of Well-posedness Condition

Below we present an equivalent form of the well-posedness condition that can be verified algebraically. By Theorem III.2 we have that if matrix  $\overline{Y} \in \mathcal{M}_r$  is well-posed, then  $\overline{Y}$  is a locally unique solution of problem (2). Note that condition (21) implies that  $\dim(\mathbb{V}_{\Omega^c}) + \dim(\mathcal{T}_{\mathcal{M}_r}(\overline{Y})) \leq n_1 n_2$ . That is, condition (21) implies that  $r(n_1 + n_2 - r) \leq m$  or equivalently  $r \leq \Re(n_1, n_2, m)$ . By Theorem III.1 we have that if  $r^* > \Re(n_1, n_2, m)$ , then the corresponding optimal solution cannot be locally unique almost surely. Note that since the space  $\mathbb{V}_{\Omega}$  is orthogonal to the space  $\mathbb{V}_{\Omega^c}$ , by duality arguments condition (21) is equivalent to the following condition

$$\mathbb{V}_{\Omega} + \mathcal{N}_{\mathcal{M}_r}(\bar{Y}) = \mathbb{R}^{n_1 \times n_2}.$$
(22)

By using formula (19) it is also possible to write condition (21) in the following form

$$\{X \in \mathbb{V}_{\Omega^c} : FXG = 0\} = \{0\},\tag{23}$$

where F is a left side complement of  $\overline{Y}$  and G is a right side complement of  $\overline{Y}$ . Recall that  $\operatorname{vec}(FXG) = (G^{\top} \otimes F)\operatorname{vec}(X)$ . Column vector of matrix  $G^{\top} \otimes F$  corresponding to component  $x_{ij}$  of vector vec(X), is  $g_j^{\top} \otimes f_i$ , where  $f_i$  is the *i*th column of matrix F and  $g_j$  is the *j*-th row of matrix G. Condition (23) means that the column vectors  $g_i^{\top} \otimes f_i$ ,  $(i, j) \in \Omega^c$ , are linearly independent. It could be noted that the left and right side complements are not unique. That is, the left side complement can be changed to QF for an arbitrary  $(n_1 - r) \times (n_1 - r)$  nonsingular matrix Q, and similarly the right side complement can be changed to GR for an arbitrary  $(n_2 - r) \times (n_2 - r)$  nonsingular matrix R. We have that  $(GR)^{\top} \otimes (QF) = (R^{\top} \otimes Q)(G^{\top} \otimes F)$ . Therefore the condition for vectors  $g_i^{\top} \otimes f_i$ ,  $(i, j) \in \Omega^c$ , to be linearly independent does not depend on a particular choice of the left and right side complements.

We obtain the following verifiable condition for checking the well-posedness of a given solution:

Theorem III.3 (Equivalent condition of well-posedness): Matrix  $\bar{Y} \in \mathcal{M}_r$  satisfies condition (21) if and only if for any left side complement F and right side complement G of  $\overline{Y}$ , the column vectors  $g_i^{\top} \otimes f_i$ ,  $(i, j) \in \Omega^c$ , are linearly independent.

A consequence of Theorem III.3 is that if  $\overline{Y} \in \mathcal{M}_r$  is wellposed, then necessarily  $(n_1 - r)(n_2 - r) \ge |\Omega^c|$ , since vectors  $g_j^{\top} \otimes f_i$  have dimension  $(n_1 - r)(n_2 - r)$ . Since  $|\Omega^c| =$  $n_1n_2 - m$ , this is equivalent to  $r(n_1 + n_2 - r) \le m$ . That is, the well-posedness cannot happen if  $r > \Re(n_1, n_2, m)$ . This of course is not surprising in view of discussion of Section III-A.

Theorem III.3 also implies the following necessary condition for well-posedness of  $\overline{Y} \in \mathcal{M}_r$  in terms of the pattern of the index set  $\Omega$ , which is related to the completability condition in [8] that each row and each column has at least r observations. If matrix  $\overline{Y} \in \mathcal{M}_r$  is well-posed for problem (2), then at each row and each column of  $\overline{Y}$  there are at least r elements of the index set  $\Omega$ . Indeed, suppose that in row  $i \in \{1, ..., n_1\}$  there are less than r elements of  $\Omega$ . This means that the set  $\sigma_i := \{j : (i, j) \in \Omega^c\}$  has cardinality greater than  $n_2 - r$ . Let F be a left side complement of  $\overline{Y}$  and G be a right side complement of  $\overline{Y}$ . Since rows  $g_j$  of G are of dimension  $1 \times (n_2 - r)$ , we have then that vectors  $g_j$ ,  $j \in \sigma_i$ , are linearly dependent, i.e.,  $\sum_{j \in \sigma_i} \lambda_j g_j = 0$ for some  $\lambda_j$ , not all of them zero. Then

$$\sum_{j \in \sigma_i} \lambda_j (g_j^\top \otimes f_i) = \left( \sum_{j \in \sigma_i} \lambda_j g_j \right)^\top \otimes f_i = 0.$$
 (24)

This contradicts the condition for vectors  $g_j^{\top} \otimes f_i$ ,  $(i, j) \in \Omega^c$ , to be linearly independent. Similar arguments can be applied to the columns of matrix  $\overline{Y}$ . This necessary condition for wellposedness is not surprising since if there is a row with less than r elements of  $\Omega$ , then this row in not uniquely defined in the corresponding rank r solution (cf., [8]). However, although necessary, the condition for the index set  $\Omega$  to have at each row and each column at least r elements is not sufficient to ensure well-posedness as shown by Theorem III.5 below. Note that by definition the matrices F and G are of full rank.

## D. Generic Nature of the Well-posedness

In a certain sense the well-posedness condition is generic, as we explain below. Denote by  $\mathcal{F}_r \subset \mathbb{R}^{n_1 \times r}$  and  $\mathcal{G}_r \subset \mathbb{R}^{n_2 \times r}$ the respective sets of matrices of rank r. Consider the set  $\Theta :=$  $\mathcal{F}_r \times \mathcal{G}_r \times \mathbb{V}_{\Omega^c}$  viewed as a subset of  $\mathbb{R}^{n_1 r + n_2 r + n_1 n_2 - m}$ , and mapping  $\mathfrak{F} : \Theta \to \mathbb{R}^{n_1 \times n_2}$  defined as

$$\mathfrak{F}(\theta) := VW^{\top} + X, \ \ \theta = (V, W, X) \in \Theta.$$

Note that the sets  $\mathcal{G}_r$  and  $\mathcal{F}_r$  are open and connected, and hence the set  $\Theta$  is open and connected, and the components of mapping  $\mathfrak{F}(\cdot)$  are polynomial functions.

Let  $\Delta(\theta)$  be the Jacobian of mapping  $\mathfrak{F}$ . That is,  $\Delta(\theta)$  is  $(n_1r + n_2r + n_1n_2 - m) \times (n_1n_2)$  matrix of partial derivatives of  $\mathfrak{F}(\theta)$  taken with respect to a specified order of the components of the corresponding matrices. Let us consider the following concept associated with rank r and index set  $\Omega$  (cf., [28]).

Definition III.3: We refer to

$$\varrho := \max_{\theta \in \Theta} \left\{ \operatorname{rank}(\Delta(\theta)) \right\}$$
(25)

as the *characteristic rank* of mapping  $\mathfrak{F}$  and say that  $\theta \in \Theta$  is a *regular point* of  $\mathfrak{F}$  if rank $(\Delta(\theta)) = \varrho$ . We say that  $(V, W) \in \mathcal{F}_r \times \mathcal{G}_r$  is regular if  $\theta = (V, W, X)$  is regular for some  $X \in \mathbb{V}_{\Omega^c}$ .

Since  $\mathfrak{F}(V, W, \cdot)$  is linear, the Jacobian  $\Delta(V, W, X)$  is the same for all  $X \in \mathbb{V}_{\Omega^c}$ , i.e.,  $\Delta(V, W, X) = \Delta(V, W, X')$  for any  $X, X' \in \mathbb{V}_{\Omega^c}$  and  $(V, W) \in \mathcal{F}_r \times \mathcal{G}_r$ . Hence if a point  $\theta = (V, W, X)$  is regular for some  $X \in \mathbb{V}_{\Omega^c}$ , then (V, W, X')is regular for any  $X' \in \mathbb{V}_{\Omega^c}$ . Therefore regularity actually is a property of points  $(V, W) \in \mathcal{F}_r \times \mathcal{G}_r$ .

Consider  $\theta = (V, W, X) \in \Theta$  and  $Y = VW^{\top}$ . We have that  $\operatorname{rank}(\Delta(\theta)) = \dim(\mathfrak{V}(\theta))$ , where  $\mathfrak{V}(\theta)$  denotes the image of the differential of  $\mathfrak{F}(\theta)$ . Since the differential  $d \mathfrak{F}(\theta) = (dV)W^{\top} + V(dW)^{\top} + dX$  and because of (18), the linear space  $\mathfrak{V}(\theta)$  is equal to  $\mathcal{T}_{\mathcal{M}_r}(Y) + \mathbb{V}_{\Omega^c}$ . It follows that  $\varrho \leq \mathfrak{f}(r, m)$ , where

$$f(r,m) := \dim(\mathcal{T}_{\mathcal{M}_r}(Y)) + \dim(\mathbb{V}_{\Omega^c}) = r(n_1 + n_2 - r) + n_1 n_2 - m.$$
(26)

It also follows that  $\operatorname{rank}(\Delta(\theta)) = \mathfrak{f}(r, m)$  iff condition (21) holds at Y. In other words we have the following result.

Proposition III.1: Rank of  $\Delta(\theta)$  attains the maximal value  $\mathfrak{f}(r,m)$  if and only if the corresponding matrix  $Y = VW^{\top}$  is well posed.

Furthermore we have the following.

Theorem III.4: The following holds: (i) Almost every point  $(V, W) \in \mathcal{F}_r \times \mathcal{G}_r$  is regular. (ii) The set of regular points forms an open subset of  $\mathcal{F}_r \times \mathcal{G}_r$ . (iii) For any regular point  $(V, W) \in \mathcal{F}_r \times \mathcal{G}_r$ , the corresponding matrix  $Y = VW^{\top}$  satisfies the well-posedness condition (21) if and only if the characteristic rank  $\varrho$  is equal to  $\mathfrak{f}(r, m)$ . (iv) If  $\varrho < \mathfrak{f}(r, m)$  and a point  $(\bar{V}, \bar{W}) \in \mathcal{F}_r \times \mathcal{G}_r$  is regular, then for any  $Y \in \mathcal{M}_r$  in a neighborhood of  $\bar{Y} = \bar{V}\bar{W}^{\top}$  there exists  $X \in \mathbb{V}_{\Omega^c}$  such that  $Y = \bar{Y} + X$ .

The significance of Theorem III.4 is that this shows that for given rank r and index set  $\Omega$ , either  $\rho = \mathfrak{f}(r, m)$  in which case a.e.  $Y \in \mathcal{M}_r$  satisfies the well-posedness condition (21), or  $\rho < \mathfrak{f}(r, m)$  in which case condition (21) does not hold for all  $Y \in \mathcal{M}_r$  and generically rank r solutions are not locally unique.

We have that a necessary condition for  $\rho = \mathfrak{f}(r,m)$  is that each row and each column of the considered matrix has at least r observed entries. Another necessary condition is for the index set to be irreducible (see Theorem III.5). Whether these two conditions are sufficient for  $\rho = \mathfrak{f}(r,m)$  to hold remains an open question. Numerical experiments, reported in Section V, indicate that in a certain probabilistic sense chances of occurring not well posed solution are negligible when r is slightly less than  $\Re(n_1, n_2, m)$ .

### E. Global Uniqueness of Solutions for Special Cases

In some rather special cases it is possible to give verifiable conditions for global uniqueness of minimum rank solutions. The following conditions are straightforward extensions of well known conditions in Factor Analysis (cf., [29, Theorem 5.1]).

Assumption III.1: Suppose that: (i) for a given index  $(k, l) \in \Omega^c$ , there exist index sets  $\mathcal{I}_1 \subset \{1, ..., n_1\} \setminus \{k\}$  and  $\mathcal{I}_2 \subset \Omega^c$ 

 $\{1, ..., n_2\} \setminus \{l\}$  such that  $|\mathcal{I}_1| = |\mathcal{I}_2| = r$ ,  $\mathcal{I}_1 \times \mathcal{I}_2 \subset \Omega$ , and  $\{k\} \times \mathcal{I}_2 \subset \Omega$  and  $\{l\} \times \mathcal{I}_1 \subset \Omega$ , (ii) the  $r \times r$  submatrix of M corresponding to rows  $i \in \mathcal{I}_1$  and columns  $j \in \mathcal{I}_2$  is nonsingular.

For example, for r = 1 part (i) of the above assumption means existence of indexes  $k' \neq k$  and  $l' \neq l$  such that  $(k', l), (k, l'), (k', l') \in \Omega$ .

Proposition III.2: Suppose that Assumption III.1 holds for an index  $(k, l) \in \Omega^c$ . Then the minimum rank  $r^* \geq r$ , and for any matrix  $Y \in \mathcal{M}_r$  such that  $P_{\Omega}(Y) = M$  it follows that  $Y_{kl} = \overline{Y}_{kl}$ .

Clearly part (ii) of Assumption III.1 implies that  $r^* \ge r$ . The other result of the above proposition follows by observing that the  $(r + 1) \times (r + 1)$  submatrix of Y corresponding to rows  $\{k\} \cup \mathcal{I}_1$  and columns  $\{l\} \cup \mathcal{I}_2$  has rank r and hence zero determinant, and applying Schur complement for the element  $Y_{kl}$ . Note that provided the part (i) holds, part (ii) is generic in the sense that it holds for a.e.  $M_{ij}$ .

If Assumption III.1 holds for every  $(k,l) \in \Omega^c$ , then the uniqueness of the solution  $\overline{Y}$  follows. This is closely related to [8, Theorem 2], but is not the same. It is assumed in [8] that every column of M has r + 1 observed entries. For example, consider  $2 \times 2$  matrix with 3 observed entries,  $M_{12} = M_{21} = M_{22} = 1$ . The only unobserved entry, corresponding to the index (1, 1), satisfies Assumption III.1 and rank one matrix, with all entries equal 1, is the unique solution of the MRMC problem. On the other hand the first column of matrix M has only one observed entry.

Remark III.1: An example was constructed in Wilson and Worcester [30], of two  $6 \times 6$  symmetric matrices of rank 3 with the same off-diagonal and different diagonal elements. If we define the index set as  $\Omega := \{(i, j) : i \neq j, i, j = 1, ..., 6\}$ , then this can be viewed as an example of two different locally unique solutions of rank 3. Note that here m = 30 and  $\Re(6, 6, 30) = 6 - \sqrt{6}$ . That is  $\Re(6, 6, 30) > 3$  and generically (almost surely) rank cannot be reduced below r = 4. We will discuss this example further in Section V.

## F. Identifiable $\Omega$

Our results can also be used to determine wether observation patterns  $\Omega$  is identifiable. First note that uniqueness of the minimum rank solution is invariant with respect to permutations of rows and columns of matrix M. This motivates to introduce the following definition.

Definition III.4: We say that the index set  $\Omega$  is *reducible* if by permutations of rows and columns, the set  $\Omega$  can be represented as the union  $\Omega' \cup \Omega''$  of two disjoined sets  $\Omega' \subset \{1, ..., k\} \times \{1, ..., l\}$  and  $\Omega'' \subset \{k + 1, ..., n_1\} \times \{l + 1, ..., n_2\}$  for some  $1 \le k < n_1$  and  $1 \le l < n_2$ . Otherwise we say that  $\Omega$  is irreducible.

Reducibility of the index set  $\Omega$  means that by permutations of rows and columns, matrix M can be represented in the block diagonal form

$$M = \begin{bmatrix} M' & 0\\ 0 & M'' \end{bmatrix},\tag{27}$$

where matrices M' and M'' are of order  $k \times l$  and  $(n_1 - k) \times (n_2 - l)$ , respectively, with observed entries  $M'_{ij}$ ,  $(i, j) \in \Omega'$ , and  $M''_{ij}$ ,  $(i, j) \in \Omega''$ . Some entries of matrices M' and M'' can also be zero if the corresponding entries of matrix M are zeros.

Theorem III.5 (Reducible index set): If the index set  $\Omega$  is reducible, then any minimum rank solution  $\overline{Y}$  is not locally (and hence globally) unique.

As it was shown in Theorem III.2, if  $\overline{Y}$  is not locally unique, then it cannot be well-posed. Therefore if the index set  $\Omega$  is reducible, then any minimum rank solution is not well-posed. Of course even if  $\Omega$  is reducible, it still can happen that in each row and column there are at least r elements of the index set  $\Omega$ . That is, the condition of having r elements of the index set  $\Omega$  in each row and column is not sufficient to ensure the well-posedness property.

Remark III.2: Reducibility/irreducibility of the index set  $\Omega$  can be verified in the following way. Consider the undirected graph G = (V, E) with the set of vertices  $V := \Omega$ , and edges between two vertices  $(i, j), (i', j') \in \Omega$  if and only if i = i' or j = j'. Then  $\Omega$  is irreducible if and only if G has only one connected component. A connected component of G is a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the supergraph G. There are algorithms of running time O(|V| + |E|) which can find every vertex that is reachable from a given vertex of G, and hence to determine a connected component of G, e.g., the well known breadth-first search algorithm [31, Section 22.2]. Note that the number of vertices in G is  $m = |\Omega|$ , which could be much smaller than  $n_1n_2$ .

### G. Uniqueness of Rank One Solutions

In this section we discuss uniqueness of rank one solutions of the MRMC problem (2). We show that in case of the minimum rank one, irreducibility of  $\Omega$  is sufficient for the global uniqueness. We assume that all  $M_{ij} \neq 0$ ,  $(i, j) \in \Omega$ , and that every row and every column of the matrix M has at least one element  $M_{ij}$ . Let  $\bar{Y}$  be a solution of rank one of problem (2), i.e., there are nonzero column vectors v and w such that  $\bar{Y} = vw^{\top}$  with  $P_{\Omega}(\bar{Y}) = M$ .

Recall that permutations of the components of vector v corresponds to permutations of the rows of the respective rank one matrix, and permutations of the components of vector w corresponds to permutations of the columns of the respective rank one matrix. It was shown in Theorem III.5 that if the index set  $\Omega$  is reducible, then solution  $\overline{Y}$  cannot be locally unique. In case of rank one solution the converse of that also holds.

Theorem III.6 (Global uniqueness for rank one solution): Suppose that  $\Omega$  is irreducible,  $M_{ij} \neq 0$  for all  $(i, j) \in \Omega$ , and every row and every column of the matrix M has at least one element  $M_{ij}$ ,  $(i, j) \in \Omega$ . Then any rank one solution is globally unique.

It could be mentioned that even for r = 1 the irreducibility is a weaker condition than part (i) of Assumption III.1 applied to every  $(k, l) \in \Omega^c$ . For example, let  $n_1 = n_2 = n \ge 3$  and  $\Omega = \{(i, j) : i \ge j, i, j = 1, ..., n\} \setminus \{(n, 1)\}$ . This set  $\Omega$  irreducible. However for the index (1, n), Assumption III.1 (i) does not hold.

#### H. Semidefinite Relaxations

Consider the trace minimization problem (9) (which can be viewed as a generalized version of the nuclear norm minimization problem), and assume that the matrix  $C \in W_{\mathfrak{S}^c}$  is *positive definite*. The (Lagrangian) dual of problem (9) is the problem

$$\max_{\Lambda \succeq 0} \min_{X \in \mathbb{W}_{\mathfrak{S}^c}} \operatorname{tr}(CX) - \operatorname{tr}[\Lambda(\Xi + X)].$$
(28)

For  $\Lambda = C - \Theta$ , with  $\Theta \in \mathbb{W}_{\mathfrak{S}}$ , problem (28) can be written (note that  $\operatorname{tr}(C\Xi) = 0$  for  $\Xi \in \mathbb{W}_{\mathfrak{S}}$ ) as

$$\max_{\Theta \in \mathbb{W}_{\mathfrak{S}}} \operatorname{tr}(\Theta \Xi) \text{ subject to } C - \Theta \succeq 0.$$
(29)

We have the following uniqueness results for the SDP approach, which is a consequence of (cf., [32, Theorem 5.2] and [14, Proposition 8]) (we also provide justification in the appendix):

Theorem III.7: (i) For a given  $\Xi \in \mathbb{W}_{\mathfrak{S}}$  it follows that for almost every positive definite matrix  $C \in \mathbb{W}_{\mathfrak{S}^c}$ , problem (9) has unique optimal solution. (ii) For a given positive definite matrix  $C \in \mathbb{W}_{\mathfrak{S}^c}$  it follows that for almost every  $\Xi \in \mathbb{W}_{\mathfrak{S}}$  the dual problem (29) has unique optimal solution.

However, we have the following observation, which comes as a consequence of [14, Theorem 2]:

*Remark III.3:* Consider the minimum trace (MT) problem (7). Suppose that the matrix  $\Xi$  is observed with errors:  $\Xi = \Xi^* + U$ , where  $U \in \mathbb{S}^p$  is random matrix such that  $N^{1/2}U$  converges in distribution to a random matrix  $\Upsilon \in \mathbb{S}^p$  whose entries have zero means and finite positive second order moments (we discuss a similar model for the MRMC in section III-I below). Let  $\hat{X}$  and  $X^*$  be optimal solutions of the MT problems of the form (7) for matrices  $\Xi$  and  $\Xi^*$ , respectively. Then under mild regularity conditions

$$\operatorname{tr}(\hat{X}) - \operatorname{tr}(X^*) = \sup_{\Lambda \in \operatorname{Sol}(D)} \operatorname{tr}(\Lambda U) + o_p(N^{-1/2}), \quad (30)$$

where  $\operatorname{Sol}(D)$  is the set of optimal solutions of the dual problem (28). When the minimal rank of the true model is less than the generic lower bound (given by the right hand side of (31)), the set  $\operatorname{Sol}(D)$  contains more than one element. Consequently  $\operatorname{tr}(\hat{X})$ , considered as an estimator of  $\operatorname{tr}(X^*)$ , has a bias of order  $N^{-1/2}\mathbb{E}[\sup_{\Lambda \in \operatorname{Sol}(D)} \operatorname{tr}(\Lambda \Upsilon)]$  (we can refer to [14, Theorem 2] for derivations and a discussion of the required regularity conditions).

We conclude this section by mentioning connections to existing results in Factor Analysis. The classical Minimum Rank Factor Analysis (MRFA) can be viewed as a particular case of problem (6) with  $\mathbb{W}_{\mathfrak{S}^c}$  being the space  $\mathbb{D}^p$  of  $p \times p$  diagonal matrices, and given symmetric matrix  $\Xi$  of off diagonal elements. It is possible to show that generically (i.e., for a.e.  $\Xi$ ) the reduced rank of the MRFA problem is bounded (cf., [33]):

$$\operatorname{rank}(\Xi + X) \ge \frac{2p + 1 - \sqrt{8p + 1}}{2}, \ \forall X \in \mathbb{D}^p.$$
(31)

In Factor Analysis the respective minimum trace problem of the form (7) is called the Minimum Trace Factor Analysis (MTFA). A relation between MRFA and MTFA problems is discussed in [33], [34]. In Factor Analysis conditions analogues to the assumptions of Proposition III.2 can be used to show that in a certain generic sense, MRFA solution is unique if the respective minimal rank is less than p/2 (we can refer to [35], and references therein, for a discussion of uniqueness of MRFA solutions).

### I. LRMA and its Properties

We discuss below the LRMA approach (5). Compared with the formulation of exact low rank recovery, the LRMA is more realistic in the presence of noise. By Theorem III.1 we have that if the minimal rank  $r^*$  is less than  $\Re(n_1, n_2, m)$ , then the corresponding solution is unstable in the sense that an arbitrary small perturbation of the observed values  $M_{ij}$  can make this rank unattainable. On the other hand if  $r^* > \Re(n_1, n_2, m)$ , then almost surely the solution is not (even locally) unique. This indicates that except in rare occasions, problem (2) of exact rank minimization cannot have both properties of possessing unique and stable solutions. Consequently, what makes sense is to try to solve the minimum rank problem approximately.

Proposition III.3 (Necessary condition for LRMA): The following are necessary conditions for  $Y \in \mathcal{M}_r$  to be an optimal solution of problem (5)

$$(P_{\Omega}(Y) - M)^{\top}Y = 0 \text{ and } Y(P_{\Omega}(Y) - M)^{\top} = 0.$$
 (32)

*Remark III.4:* We can view the least squares problem (5) from the following point of view. Consider function

$$\phi(Y,\Theta) := \frac{1}{2} \operatorname{tr}[(P_{\Omega}(Y) - \Theta)^{\top}(P_{\Omega}(Y) - \Theta)], \qquad (33)$$

with  $\Theta \in \mathbb{V}_{\Omega}$  viewed as a parameter. Define

$$f(Y) := \frac{1}{2} \sum_{(i,j)\in\Omega} (Y_{ij} - M_{ij})^2$$
  
=  $\frac{1}{2} \operatorname{tr}[(P_{\Omega}(Y) - M)^{\top}(P_{\Omega}(Y) - M)], \quad (34)$ 

Hence, the problem (5) consists of minimization of f(Y) subject to  $Y \in \mathcal{M}_r$ . Note that for  $\Theta = M$  we have  $f(\cdot) = \phi(\cdot, M)$ , where  $f(\cdot)$  is defined in (34). Let  $\overline{Y} \in \mathcal{M}_r$  be such that  $\phi(\overline{Y}, \Theta_0) = 0$  for some  $\Theta_0 \in \mathbb{V}_\Omega$ , i.e.,  $P_\Omega(\overline{Y}) = \Theta_0$ . A sufficient condition for  $\overline{Y}$  to be a locally unique solution of problem (2), at  $M = \Theta_0$ , is

$$\operatorname{tr}\left[P_{\Omega}(H)^{\top}P_{\Omega}(H)\right] > 0, \quad \forall H \in \mathcal{T}_{\mathcal{M}_{r}}(\bar{Y}) \setminus \{0\}.$$
(35)

The above condition means that if  $H \in \mathcal{T}_{\mathcal{M}_r}(\overline{Y})$  and  $H \neq 0$ , then  $P_{\Omega}(H) \neq 0$ . In other words this means that the kernel

$$\operatorname{Ker}(P_{\Omega}) := \{ H \in \mathcal{T}_{\mathcal{M}_r}(\bar{Y}) : P_{\Omega}(H) = 0 \}$$

is {0}. Since  $P_{\Omega}(H) = 0$  for any  $H \in \mathbb{V}_{\Omega^c}$ , it follows that: condition (35) is equivalent to the sufficient condition (21) of *Proposition* III.2. That is, condition (35) means that matrix  $\overline{Y}$  is well-posed for problem (2). Assuming that condition (35) (or equivalently condition (21)) holds, by applying the Implicit Function Theorem to the first order optimality conditions of the least squares problem (5) we have the following result.

Proposition III.4: Let  $\bar{Y} \in \mathcal{M}_r$  be such that  $P_{\Omega}(\bar{Y}) = \Theta_0$ for some  $\Theta_0 \in \mathbb{V}_{\Omega}$  and suppose that the well posedness condition (21) holds. Then there exist neighborhoods  $\mathcal{V}$  and  $\mathcal{W}$  of  $\bar{Y}$  and  $\Theta_0$ , respectively, such that for any  $M \in \mathcal{W} \cap \mathbb{V}_{\Omega}$  there exists unique  $Y \in \mathcal{V} \cap \mathcal{M}_r$  satisfying the optimality conditions (32).

The above proposition implies the following. Suppose that we run a numerical procedure which identifies a matrix  $\bar{Y} \in \mathcal{M}_r$  satisfying the (necessary) first order optimality conditions (32). Then if  $P_{\Omega}(\bar{Y})$  is sufficiently close to M (i.e., the fit  $\sum_{(i,j)\in\Omega} (Y_{ij} - M_{ij})^2$  is sufficiently small) and condition (21) holds at  $\bar{Y}$ , then we can say that  $f(Y) > f(\bar{Y})$  for all  $Y \neq \bar{Y}$  in a neighborhood of  $\bar{Y}$ . That is,  $\bar{Y}$  solves the least squares problem at least locally. Unfortunately it is not clear how to quantify the "sufficiently close" condition, and this does not guarantee global optimality of  $\bar{Y}$  unless  $\bar{Y}$  is the unique minimum rank solution.

## IV. STATISTICAL TEST FOR RANK SELECTION

In this section, we propose a statistical test procedure for value of the "true" minimal rank, when the entries of the data matrix M are observed with noise. Such statistical approach can be useful for many existing low-rank matrix completion algorithms, which require a pre-specification of the matrix rank, such as the alternating minimization approach to solving the non-convex problem by representing the low-rank matrix as a product of two low-rank matrix factors (see, e.g., [4]).

Consider the LRMA formulation. By the above discussion, it will be natural to take some value of r less than  $\Re(n_1, n_2, m)$ , since otherwise we will not even have locally unique solution. Can the fit of  $Y \in \mathcal{M}_r$  to X + M, and hence the choice of r, be tested in some statistical sense?

To proceed we assume the following model with noisy and possibly biased observations of a subset of matrix entries. There is a (population) value  $Y^*$  of  $n_1 \times n_2$  matrix of rank  $r < \Re(n_1, n_2, m)$  and  $M_{ij}$  are viewed as observed (estimated) values of  $Y_{ij}^*$ ,  $(i, j) \in \Omega$ , based on a sample of size N. The observed values are modeled as

$$M_{ij} = Y_{ij}^* + N^{-1/2} \Delta_{ij} + \varepsilon_{ij}, \ (i,j) \in \Omega, \tag{36}$$

where  $Y^* \in \mathcal{M}_r$  and  $\Delta_{ij}$  are some (deterministic) numbers. The random errors  $\varepsilon_{ij}$  are assumed to be independent of each other and such that  $N^{1/2}\varepsilon_{ij}$  converge in distribution to normal with mean zero and variance  $\sigma_{ij}^2$ ,  $(i, j) \in \Omega$ . The additional terms  $N^{-1/2}\Delta_{ij}$  in (36) represent a possible deviation of population values from the "true" model and are often referred to as the population drift or a sequence of local alternatives (we can refer to [36] for a historical overview of invention of the local alternatives setting). This is a reasonably realistic model motivated by many real applications.

Definition IV.1: We say that the model is globally identifiable (at  $Y^*$ ) if  $\overline{Y} \in \mathbb{R}^{n_1 \times n_2}$  of rank $(\overline{Y}) \leq r$  and  $P_{\Omega}(\overline{Y}) = P_{\Omega}(Y^*)$ imply that  $\overline{Y} = Y^*$ , i.e.,  $Y^*$  is the unique solution of the respective matrix completion problem. Similarly it is said that the model is *locally identifiable* if this holds for all such  $\overline{Y}$  in a neighborhood of  $Y^*$ , i.e.,  $Y^*$  is a locally unique solution.

Consider the following weighted least squares problem (a generalization of (5)):

$$\min_{Y \in \mathcal{M}_r} \sum_{(i,j) \in \Omega} w_{ij} \left( M_{ij} - Y_{ij} \right)^2, \tag{37}$$

for some weights  $w_{ij} > 0$ ,  $(i, j) \in \Omega$ . (Of course, if  $w_{ij} = 1$ ,  $(i, j) \in \Omega$ , then problem (37) coincides with the least squares problem (5)). We have the following standard result about consistency of the least squares estimates.

Proposition IV.1: Suppose that the model is globally identifiable at  $Y^* \in \mathcal{M}_r$  and values  $M_{ij}$ ,  $(i, j) \in \Omega$ , converge in probability to the respective values  $Y_{ij}^*$  as the sample size Ntends to infinity. Then an optimal solution  $\hat{Y}$  of problem (37) converges in probability to  $Y^*$  as  $N \to \infty$ .

Consider the following weighted least squares test statistic

$$T_N(r) := N \min_{Y \in \mathcal{M}_r} \sum_{(i,j) \in \Omega} w_{ij} \left( M_{ij} - Y_{ij} \right)^2, \quad (38)$$

where  $w_{ij} := 1/\hat{\sigma}_{ij}^2$  with  $\hat{\sigma}_{ij}^2$  being consistent estimates of  $\sigma_{ij}^2$ (i.e.,  $\hat{\sigma}_{ij}^2$  converge in probability to  $\sigma_{ij}^2$  as  $N \to \infty$ ). Recall that the respective condition of form (21), or equivalently (35), is sufficient for local identifiability of  $Y^*$ . The following asymptotic results can be compared with similar results in the analysis of covariance structures (cf., [37]).

Proposition IV.2 (Asymptotic properties of test statistic): Consider the noisy observation model (36). Suppose that the model is globally identifiable at  $Y^* \in \mathcal{M}_r$  and  $Y^*$  is well-posed for problem (2). Then as  $N \to \infty$ , the test statistic  $T_N(r)$  converges in distribution to noncentral  $\chi^2$  distribution with degrees of freedom  $df_r = m - r(n_1 + n_2 - r)$  and the noncentrality parameter

$$\delta_r = \min_{H \in \mathcal{T}_{\mathcal{M}_r}(Y^*)} \sum_{(i,j) \in \Omega} \sigma_{ij}^{-2} \left( \Delta_{ij} - H_{ij} \right)^2.$$
(39)

Note that the optimal (minimal) value of the weighted least squares problem (37) can be approximated by

$$\min_{H \in \mathcal{T}_{\mathcal{M}_{r}}(Y^{*})} \sum_{(i,j) \in \Omega} w_{ij} \left( E_{ij} - H_{ij} \right)^{2} + R_{N}, \qquad (40)$$

with  $E_{ij} := N^{-1/2} \Delta_{ij} + \varepsilon_{ij}$  and the error term  $R_N = o\left(\|M - P_{\Omega}(Y^*)\|^2\right)$  being of stochastic order  $R_N = o_p(N^{-1})$ . Hence, the noncentrality parameter, given in (39), can be approximated as

$$\delta_r \approx N \min_{Y \in \mathcal{M}_r} \sum_{(i,j) \in \Omega} w_{ij} \left( Y_{ij}^* + N^{-1/2} \Delta_{ij} - Y_{ij} \right)^2.$$
(41)

That is, the noncentrality parameter is approximately equal to N times the fit to the "true" model of the alternative population values  $Y_{ij}^* + N^{-1/2}\Delta_{ij}$  under small perturbations of order  $O(N^{-1/2})$ .

*Remark IV.1:* The above asymptotic results are formulated in terms of the "sample size N" suggesting that the observed values are estimated from some data. That is, the given values  $\overline{M}_{ij}, (i, j) \in \Omega$ , are obtained by averaging i.i.d. data points  $M_{ij}^{\ell}$ ,  $\ell = 1, ..., N$ . In that case asymptotic normality of  $N^{1/2} \varepsilon_{ij}$  can be justified by application of the Central Limit Theorem, and the corresponding variances  $\sigma_{ij}^2$  can be estimated from the data in the usual way  $\hat{\sigma}_{ij}^2 = (N-1)^{-1} \sum_{\ell=1}^N (M_{\ell j}^{\ell} - \bar{M}_{ij})^2$ . This model allows to formulate mathematically precise convergence results. One can take a more pragmatic point of view that when there is a "small" random noise in the observed values, the respective test statistics properly normalized with respect to magnitude of that noise have approximately a noncentral chi square distribution.

The asymptotics of the test statistic  $T_N(r)$  depends on r and also on the cardinality m of the index set  $\Omega$ . Suppose now that more observations become available at additional entries of the matrix. That is we are testing now the model with a larger index set  $\Omega'$ , of cardinality m', such that  $\Omega \subset \Omega'$ . In order to emphasize that the test statistic also depends on the corresponding index set we add the index set in the respective notations. Note that if  $Y^*$  is a solution of rank r for both sets  $\Omega$  and  $\Omega'$  and the model is globally (locally) identifiable at  $Y^*$  for the set  $\Omega$ , then the model is globally (locally) identifiable at  $Y^*$  for the set  $\Omega'$ . Note also that if the regularity condition (21) holds at  $Y^*$  for the smaller model (i.e. for  $\Omega$ ), then it holds at  $Y^*$  for the larger model (i.e. for  $\Omega'$ ). The following result can be proved in the same way as Theorem IV.2 (cf., [37]).

Proposition IV.3: Consider index sets  $\Omega \subset \Omega'$  of cardinality  $m = |\Omega|$  and  $m' = |\Omega'|$ , and the noisy observation model (36). Suppose that the model is globally identifiable at  $Y^* \in \mathcal{M}_r$  and condition (21) holds at  $Y^*$  for the smaller model (and hence for both models). Then the statistic  $T_N(r, \Omega') - T_N(r, \Omega)$  converges in distribution to noncentral  $\chi^2$  with  $df_{r,\Omega'} - df_{r,\Omega} = m' - m$  degrees of freedom and the noncentrality parameter  $\delta_{r,\Omega'} - \delta_{r,\Omega}$ , and  $T_N(r, \Omega') - T_N(r, \Omega)$  is asymptotically independent of  $T_N(r, \Omega)$ .

For given index set  $\Omega$  and observed (estimated) values  $M_{ij}$ ,  $(i, j) \in \Omega$ , the statistic  $T_N(r)$  can be used for testing the (null) hypothesis that the "true" rank is r. That is the null hypothesis is rejected if  $T_N(r)$  is large enough on the scale of the  $\chi^2$  distribution with the respective df<sub>r</sub> degrees of freedom. It is often observed in practice that such tests reject the null hypothesis even when the fit is reasonable. In that respect the role of values  $\Delta_{ij}$  in the model is to suggest that the "true" model is true only approximately, and the corresponding noncentrality parameter  $\delta_r$  gives an indication of the deviation from the exact rank rmodel. It is a common practice to perform such tests sequentially for increasing values of r, with all deficiencies of such sequential testing.

Such testing procedure assumes that the sample size N is given and the corresponding variances  $\sigma_{ij}^2$  can be consistently estimated. When the observed values are obtained by averaging N data points, this is available in the straightforward way (see Remark IV.1). Otherwise setting N = 1 and assuming that all  $\sigma_{ij}^2 = \sigma^2$ ,  $(i, j) \in \Omega$ , are equal to each other, we need to specify range of  $\sigma^2$ . We will discuss this further in Section V.

*Remark IV.2:* It is also possible to give asymptotic distribution of solutions of problem (37). Suppose now that the assumptions of Theorem IV.2 hold with all  $\Delta_{ij}$  in equation (36) being zeros. Let  $\hat{Y}_N$  be a solution of problem (37), i.e.,

$$\hat{Y}_N \in \arg\min_{Y \in \mathcal{M}_r} \sum_{(i,j) \in \Omega} w_{ij} \left( \underbrace{Y_{ij}^* + \varepsilon_{ij}}_{M_{ij}} - Y_{ij} \right)^2.$$
(42)

Consider operator  $\mathcal{A} : \mathbb{V}_{\Omega} \to \mathcal{T}_{\mathcal{M}_r}(Y^*)$  defined as

$$\mathcal{A}(W) := \arg \min_{H \in \mathcal{T}_{\mathcal{M}_{r}}(Y^{*})} \sum_{(i,j) \in \Omega} \sigma_{ij}^{-2} \left( W_{ij} - H_{ij} \right)^{2}, \quad (43)$$

for  $W \in \mathbb{V}_{\Omega}$ . Because of the assumption of well posedness (which is equivalent to (35)) the minimizer in (43) is unique and hence  $\mathcal{A}(W)$  is well defined. Then

$$\hat{Y}_N = \mathcal{A}(M) + o_p(N^{-1/2}).$$
 (44)

Note that the operator  $\mathcal{A}$  is linear.

We have that  $Y^* \in \mathcal{T}_{\mathcal{M}_r}(Y^*)$  and hence  $\mathcal{A}(P_\Omega(Y^*)) = Y^*$ . Thus  $\mathcal{A}(M) = Y^* + \mathcal{A}(E)$ , where  $E \in \mathbb{R}^{n_1 \times n_2}$  is such that  $E_{ij} = \varepsilon_{ij}$  for  $(i, j) \in \Omega$ , and  $E_{ij} = 0$  otherwise. Since  $N^{1/2}\varepsilon_{ij}$ ,  $(i, j) \in \Omega$ , converge in distribution to normal with mean zero and variance  $\sigma_{ij}^2$  and independent of each over, it follows that  $N^{1/2}(\hat{Y}_N - Y^*)$  converges in distribution to the random matrix  $\mathcal{A}(Z)$ , where  $Z \in \mathbb{V}_\Omega$  is a random matrix with entries  $Z_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2), (i, j) \in \Omega$ , having normal distribution and independent of each over. Note that since  $\mathcal{A}(\cdot)$  is a linear operator,  $\mathcal{A}(Z)$  has a multivariate normal distribution with zero means. Since  $\mathcal{A}(Z)$  belongs to the linear subspace  $\mathcal{T}_{\mathcal{M}_r}(Y^*)$  of  $\mathbb{R}^{n_1 \times n_2}$ , the multivariate normal distribution of  $\mathcal{A}(Z)$  is degenerate.

## V. NUMERICAL EXAMPLES

We present some numerical experiments to illustrate our theory.<sup>1</sup> In this section, without further notification, the nuclear norm minimization is solved by TFOCS [38] in Matlab and LRMA problem is solved by "SoftImpute" [39] (regularization parameter equals to 0) in R.

## A. An Example of $6 \times 6$ Matrix Considered in [30]

As pointed in Remark III.1, Wilson and Worcester showed in [30] using analysis that there are two different locally unique solutions of rank  $r^* = 3$  for a  $6 \times 6$  matrix with the index set  $\Omega$  corresponding to its off-diagonal elements. The matrix M in that example is given by

$$M = \begin{pmatrix} 0 & 0.56 & 0.16 & 0.48 & 0.24 & 0.64 \\ 0.56 & 0 & 0.20 & 0.66 & 0.51 & 0.86 \\ 0.16 & 0.20 & 0 & 0.18 & 0.07 & 0.23 \\ 0.48 & 0.66 & 0.18 & 0 & 0.3 & 0.72 \\ 0.24 & 0.51 & 0.07 & 0.30 & 0 & 0.41 \\ 0.64 & 0.86 & 0.23 & 0.72 & 0.41 & 0 \end{pmatrix}$$

It can be verified that there are two rank **3** solutions by filling the diagonal entries by (0.64, 0.85, 0.06, 0.56, 0.50, 0.93), and (0.42, 0.90, 0.06, 0.55, 0.39, 1.00), respectively.

<sup>1</sup>More discussions can be found in a supplementary material at https:// www2.isye.gatech.edu/~yxie77/Experiment.pdf 1098



Fig. 2. Probability that well-posedness is satisfied; random instances for different rank and sampling probability. For each sampling probability and rank, we generate  $Y^*$  and  $\Omega$ . Then, we check the well-posedness condition and compute the probability. Blue curve is the estimated generic bound for the corresponding sampling probability.

This simple test case where we know the ground truth can illustrate the problem. Both the nuclear norm minimization and LRMA fail to recover any of these two local solutions above. The soft-thresholded SVD converges to a completely incorrect solution with off-diagonals far off from those of M, and the nuclear norm minimization produces a rank 4 solution by filling out the diagonal entries by (0.44, 0.76, 0.05, 0.53, 0.19, 0.96). Note that here both optimal solutions satisfy the well-posedness condition, and yet these numerical procedures can not recover either one of them. It is not clear how typical this example, of different locally optimal solutions, is. Recall that generically the nuclear norm minimization problem possesses unique optimal solution. However, it is not clear how well it approximates the "true" minimal rank solution when there is observation noise.

#### B. Probability of Well-posedness

We show the probability of satisfying the well-posedness condition by generating random cases. For each rank  $r^*$ , we generate an  $40 \times r^*$  orthonormal matrix V, an  $50 \times r^*$  orthonormal matrix W, and an  $r^* \times r^*$  diagonal matrix D. Set  $Y^* = VDW^{\top}$ . For each instance, we randomly generate the observation pattern  $\Omega$  such that each entry is observed with probability p. We check the well-posedness condition according to Theorem III.3 and using the verifiable algebraic condition. Then we repeat the above procedure 100 times and compute the percentage of cases that satisfy the well-posedness condition. Figure 2 shows the resulted proportion. We also plotted the generic bound  $\Re(n_1, n_2, p) =$  $(n_1 + n_2)/2 - ((n_1 + n_2)^2/4 - n_1n_2p)^{1/2}$ . Figure 2 shows that the probability that a matrix satisfies the well-posedness condition is not small, when the true rank is less than the generic lower bound. Moreover, the probability converge to 1 quickly, when the rank is 2 or 3 less than the generic bound. This demonstrates that the  $\Re(n_1, n_2, p)$  is a sharp bound.



Fig. 3. When the well-posedness condition is satisfied, the absolute errors at each entries  $|Y_{ij} - Y^*_{ij}|$  for the LRMA (middle panel) and the nuclear norm minimization (right panel). The left panel shows the sampling pattern  $\Omega$ . Here the true matrix  $Y^* \in \mathbb{R}^{40 \times 50}$ ,  $\mathrm{rank}(Y^*) = 10$ ,  $|\Omega| = 1000$ ,  $\varepsilon_{ij} \sim N(0, 5^2/50)$  and the observation matrix  $M_{ij} = Y^*_{ij} + \varepsilon_{ij}$ ,  $(i, j) \in \Omega$ .



Fig. 4. When the well-posedness condition is violated, the absolute errors at each entries  $|Y_{ij} - Y^*_{ij}|$  for the LRMA (middle panel) and the nuclear norm minimization (right panel). The left panel shows the sampling pattern  $\Omega$ . Here the true matrix  $Y^* \in \mathbb{R}^{70 \times 40}$ ,  $\operatorname{rank}(Y^*) = 11$ ,  $|\Omega| = 1300$ ,  $\varepsilon \sim N(0, 5^2/50)$  and the observation matrix  $M_{ij} = Y^*_{ij} + \varepsilon_{ij}$ ,  $(i, j) \in \Omega$ . The necessary condition for the well-posedness condition is violated (i.e., the numbers of observations are less than 11) at row with index numbers 3, 6, 30, 46, 50.

## C. Comparison of LRMA and Nuclear Norm Minimization

In this section, we compare the performance of LRMA and matrix completion using standard nuclear norm minimization, when the well-posedness condition is satisfied and when it is violated, respectively. The results show that the well-posedness condition is indeed necessary for good recovery performance. Moreover, our examples show that LRMA performs more stable than nuclear norm minimization in these cases.

We generate  $Y^*$ , an  $n_1 \times n_2$  matrix of rank  $r^*$ , by uniformly generated an  $n_1 \times r^*$  matrix V, an  $n_2 \times r^*$  matrix Wand an  $r^* \times r^*$  diagonal matrix D and setting  $Y^* = \tilde{V}D\tilde{W}^{\top}$ , where  $\tilde{V}$  and  $\tilde{W}$  are orthonormalization of V, W, respectively. We again sample  $\Omega$  uniformly random with probability p, where  $|\Omega| = m$ . Observation matrix M is generated by  $M_{ij} = Y_{ij}^* + \varepsilon_{ij}, (i, j) \in \Omega$ , where  $\varepsilon_{ij} \sim N(0, \sigma^2 N^{-1})$ . Algorithms stop when either relative change in the Frobenius norm between two successive estimates,  $||Y^{(t+1)} - Y^t||_F / ||Y^{(t)}||_F$ , is less than some tolerance, denoted as tol or the number of iterations exceeds the maximum it.

1) Element-wise error for three cases: We first consider three individual instances, when the well-posedness condition is satisfied and violated, respectively:

(1) In Figure 3 the well-posedness condition is satisfied. The element-wise reconstruction error for LRMA is much smaller than that of the nuclear norm minimization. In this experiment,  $n_1 = 40$ ,  $n_2 = 50$ ,  $r^* = 10$ , m = 1000,  $\sigma = 5$ , N = 50 and  $\Omega$  is sampled until the well-posedness condition is satisfied. The parameters are  $tol = 10^{-20}$  and it = 50000.



Fig. 5. When  $\Omega$  is reducible, the absolute errors at each entries  $|Y_{ij} - Y^*_{ij}|$  for the LRMA (middle panel) and the nuclear norm minimization (right panel). The left panel shows the sampling pattern  $\Omega$ . Here the true matrix  $Y^* \in \mathbb{R}^{40 \times 50}$ , rank $(Y^*) = 10$ ,  $|\Omega| = 1000$ ,  $\varepsilon_{ij} \sim N(0, \frac{5^2}{50})$  and the observation matrix  $M_{ij} = Y^*_{ij} + \varepsilon_{ij}$ ,  $(i, j) \in \Omega$ .  $\Omega$  is reducible. In this case, only two diagonal block matrices  $M_1 \in \mathbb{R}^{20 \times 20}$  and  $M_2 \in \mathbb{R}^{20 \times 30}$  are observed.

(2) In Figure 4, the well-posedness condition is violated. As predicted by our theory, both LRMA and nuclear perform worse, and the errors are especially large at index numbers 3, 6, 30, 46, 50, where the necessary condition for the well-posedness condition is violated. Still, in this situation, the nuclear norm minimization has a larger total recover error than LRMA. In this experiment,  $n_1 = 70, n_2 = 40, r^* = 11, m = 1300, \sigma = 5$ , and N = 50. We repeatedly sample  $\Omega$  until the necessary condition for the well-posedness condition is violated. The parameters  $tol = 10^{-16}$  and it = 50000. (3) In Figure 5,  $\Omega$  is reducible and thus the well-posedness condition is violated. Consistent with our theory, in this situation, both methods fail to recover the true matrix since the necessary condition of local uniqueness is violated. In this experiment,  $n_1 = 40$ ,  $n_2 =$ 50,  $r^* = 10$ , m = 1000,  $\sigma = 5$ , N = 50 and  $\Omega = \{(i, j) \in$  $\{1 \cdots 20\} \times \{1 \cdots 20\} \cup \{21 \cdots 40\} \times \{21 \cdots 50\}\}$ . The parameters are  $tol = 10^{-20}$  and it = 50000.

2) *Mean-square-error performance:* In this section, we consider the mean-square-error performance, defined by

$$MSE = \frac{1}{n_1 n_2 K} \sum_{k=1}^{K} \sum_{i,j} (Y_{ij,k}^* - \hat{Y}_{ij,k})^2$$

where K is the total number of repetitions. Figure 6 shows the difference between the *mean square error* of LRMA and the nuclear norm minimization. In this experiment,  $n_1 = 40$ ,  $n_2 = 50$ ,  $\sigma = 5$ , and we generate 50 random instances to compute the average error. The estimated  $\hat{\Re}(n_1, n_2, p)$  is also drawn as the blue curve. Figure 6 shows that, indeed, as predicted by our theory, when the true rank is lower than the generic lower bound, the performance of LRMA is much better than that of the nuclear norm minimization.

## D. Testing for True Rank

1) Asymptotic distribution of test statistic: In Section IV (see (36)), we show that the asymptotical distribution of the test statistic for the "true" rank is  $\chi^2$  distribution, which we will verify numerically here. We generate the true matrix  $Y^*$ , an  $n_1 \times n_2$  matrix of rank  $r^*$ , by uniformly generated an  $n_1 \times r^*$  matrix V, an  $n_2 \times r^*$  matrix W, and an  $r^* \times r^*$  diagonal matrix D and setting  $Y^* = \tilde{V}D\tilde{W}^{\top}$ , where  $\tilde{V}$  and  $\tilde{W}$  are orthonormalization of V, W, respectively. We sample  $\Omega$  uniformly ran-



Fig. 6. Difference between the MSEs of LRMA and the nuclear norm minimization. The blue curve is the generic bound for the corresponding sampling probability.



Fig. 7. Q-Q plot of  $T_N(r)$  against quantiles of  $\chi^2$  distribution:  $Y^* \in \mathbb{R}^{40 \times 50}$ ,  $rank(Y^*) = 11$ ,  $|\Omega| = 1000$ , the observation matrix M is generated 200 times,  $M_{ij}^{(k)} = Y_{ij}^* + \varepsilon_{ij}^{(k)}$ ,  $(i, j) \in \Omega$ , where  $\varepsilon_{ij}^{(k)} \sim N(0, 5^2/400)$ . For each  $M^{(k)}$ ,  $T_N^{(k)}(r)$  is computed as equation 38. By Theorem IV.2,  $\{T_N^{(k)}(r)\}$  follows central  $\chi^2$  distribution with the degree-of-freedom df $_r = m - r(n_1 + n_2 - r) = 131$ .

dom, where  $|\Omega| = m$ . The noisy and repeated observation matrices are generated by  $M_{ij}^{(k)} = Y_{ij}^* + \varepsilon_{ij}^{(k)}, (i, j) \in \Omega$ , where  $\varepsilon_{ij}^{(k)} \sim N(0, \sigma^2 N^{-1})$ . When computing the test statistic  $T_N^{(k)}(r)$  (38), the least square approximation problem is solved by the soft-thresholded SVD solver. The algorithm stops when either relative change in the Frobenius norm between two successive estimates is less than some tolerance, denoted as *tol*, or the number of iterations reaches the maximum, denoted as *it*.

Figure 7 shows the Q-Q plot of  $\{T_N^{(k)}(r)\}_{k=1}^{200}$  against the corresponding  $\chi^2$  distribution. In this experiment,  $n_1 = 40$ ,  $n_2 = 50$ ,  $r^* = 11$ , m = 1000,  $\sigma = 5$ , N = 400 and  $\Omega$  is sampled until the well-posedness condition is satisfied. The pa-

Fig. 8. Q-Q plot of  $T_N(r, \Omega') - T_N(r, \Omega)$  against the quantiles of  $\chi^2$  distribution:  $Y^* \in \mathbb{R}^{40 \times 50}$ , rank $(Y^*) = 11$ ,  $|\Omega'| = 1001$ ,  $|\Omega| = 996$ , where  $\Omega \subset \Omega'$ . The observation matrix M' and M are generated 200 times, By Theorem IV.3,  $\{T_N^{(k)}(r, \Omega') - T_N^{(k)}(r, \Omega)\}$  follows central  $\chi^2$  distribution with the degree-of-freedom  $\mathrm{df}_{r,\Omega'} - \mathrm{df}_{r,\Omega} = m' - m = 5$ .

N = 50 m' = 1001 m = 996

 TABLE I

 p-value for Sequential Rank Test in Simulation

rank

p-value

rank

2	0.00	8	0.00
3	0.00	9	0.94
4	0.00	10	0.69
5	0.00	11	0.41
6	0.00	12	0.00

rameters  $tol = 10^{-20}$  and it = 50000. From the result, we can see  $T_N(r)$  follows a central  $\chi^2$  distribution with a degree-offreedom df<sub>r</sub> =  $m - r(n_1 + n_2 - r) = 131$ , which is consistent with Theorem IV.2.

Figure 8 shows the Q-Q plot of  $\{T_N^{(k)}(r,\Omega') - T_N^{(k)}(r,\Omega)\}_{k=1}^{200}$  against the corresponding  $\chi^2$  distribution. In this experiment,  $n_1 = 40$ ,  $n_2 = 50$ ,  $r^* = 11$ , m = 996,  $\sigma = 5$ , N = 50,  $m' = |\Omega'| = 1001$  and  $\Omega$  is sampled until the well-posedness condition is satisfied. Note that  $\Omega'$  also satisfied well-posedness condition since  $\Omega'^C \subset \Omega^C$ . The parameters  $tol = 10^{-20}$  and it = 50000. From the result, we can see  $T_N(r,\Omega') - T_N(r,\Omega)$  follows a central  $\chi^2$  distribution with a degree-of-freedom  $df_{r,\Omega'} - df_{r,\Omega} = m' - m = 5$ , which is consistent with Theorem IV.3.

2) Test for true rank: As discussed in Section IV, we can determine the true rank  $r^*$  by sequential  $\chi^2$  tests. That is, for r ranging from 1 to  $[\Re(n_1, n_2, m)]$ , we solve the least square approximations and compute  $T_N(r)$ . According to  $T_N(r)$  we can determine which rank can be accepted for a predefined significant level. Table I shows a result of sequential rank test on a simulated data set. In this experiment,  $n_1 = 40$ ,  $n_2 = 50$ ,  $r^* = 9$ , m = 1000,  $\sigma = 5$ , N = 100, and  $\Omega$  is sampled until well-posedness condition is satisfied. The true rank 9, is the first one accepted for 0.05 significant level.

Fig. 9. Comparison of rank selection between sequential  $\chi^2$  test, the nuclear norm minimization and the  $M^E$  method, when the sampling probability p=0.3. For each true rank, we compute the median of rank error for 100 experiments.  $Y^{*(k)} \in \mathbb{R}^{100 \times 1000}$ ,  $M_{ij}^{(k)} = Y_{ij}^{*(k)} + \varepsilon_{ij}^{(k)}$ ,  $(i, j) \in \Omega$ , where  $\varepsilon_{ij}^{(k)} \sim N(0, 5^2/50)$ . Threshold  $b_{nm} = 0.25$ ,  $b_{ME} = 0.13$  for the nuclear norm minimization and the  $M^E$  method, respectively.

true rank r\*

Ŕ

ġ

10

Ż

Ż

Figure 9 shows the comparison of rank selection between our sequential rank test, the nuclear norm minimization and the method suggested in [40] (we refer to it as  $M^E$ method in the following). Since the nuclear norm minimization and the  $M^E$  method cannot give the exact rank, we choose the rank by thresholding the percentage of the singular value of the recovered matrix in this two methods, i.e.  $\hat{r} = \operatorname{argmin}_r \sum_{i=1}^r \lambda_{(i)} / \sum_{i=1}^{min(n_1, n_2)} \lambda_{(i)} > b$ , where b is a chosen threshold. In this experiment,  $n_1 = 100, n_2 = 1000$ ,  $\sigma = 5, N = 50$  and the sampling probability p = 0.3. For each true rank, we generate 100 instances of  $(Y^*, \Omega, M)$ , complete the rank selection with these three methods and compute the median of the error of estimated rank of each method. For the sequential rank test, we choose the first rank accepted with a 0.05 significant level. For the nuclear norm minimization and the  $M^E$  method, we choose the threshold that gives us the best results for these two methods. It shows that selection by sequential  $\chi^2$  test outperforms the other two methods.

#### VI. CONCLUSION

In this paper, we have examined the matrix completion from a geometric viewpoint and established a sufficient condition for local uniqueness of solutions. Our characterization assumes deterministic patterns and the results are general. We argue that the exact minimum rank matrix completion (MRMC) leads to either unstable or non-unique solutions and thus the alternative low-rank matrix approximation (LRMA) is a more reasonable approach. We propose a statistical test for rank selection, based on observed entries, which can be useful for practical matrix completion algorithms. Assuming the model (36), it is also pos-



p-value

0

20

5

9

ß

o ⊮ 0

Inr' – Tnr statistics quantiles

sible to derive asymptotics of the optimal value and, under rather stringent conditions, of the optimal solutions of the minimum trace (MT) problem (7) (cf., [14]).

For small values of the "true" rank, when the respective dual of the "true" MT problem has more than one optimal solution, the asymptotic bias of the optimal value of the approximating MT problem is of order  $O(N^{-1/2})$  (see Remark III.3). On the other hand, under the model (36) when the values  $M_{ij}$ ,  $(i, j) \in \Omega$ , are computed by averaging N data points having normal distribution (see Remark IV.1), the least squares approach corresponds to the Maximum Likelihood method which is an asymptotically efficient estimation procedure. This gives an insight into the relatively poor performance of the nuclear norm approach, as compared with the least squares method, as reported in Section V.

#### APPENDIX

Proof of Theorem III.2: We argue by a contradiction. Suppose that there is a sequence  $\{Y_k\} \subset \mathcal{M}_r$  (with  $Y_k \neq \overline{Y}$ ) converging to  $\overline{Y}$  such that  $P_{\Omega}(Y_k) = M$ . It follows that  $Y_k - \overline{Y} \in \mathbb{V}_{\Omega^c}$ . By passing to a subsequence if necessary we can assume that  $(Y_k - \overline{Y})/t_k$ , where  $t_k := ||Y_k - \overline{Y}||$ , converges to some  $H \in \mathbb{V}_{\Omega^c}$ . Note that  $H \neq 0$ . Moreover  $Y_k = \overline{Y} + t_k H + o(t_k)$ , and hence  $H \in \mathcal{T}_{\mathcal{M}_r}(\overline{Y})$ . That is  $H \in \mathbb{V}_{\Omega^c} \cap \mathcal{T}_{\mathcal{M}_r}(\overline{Y})$ , and  $H \neq 0$  by the construction. This gives the desired contradiction with (21).

Proof of Theorem III.4: Let  $\varrho$  be the characteristic rank of mapping  $\mathfrak{F}$ . Consider  $\theta^* \in \Theta$  such that  $\varrho = \operatorname{rank}(\Delta(\theta^*))$ . It follows that matrix  $\Delta(\theta^*)$  has an  $\varrho \times \varrho$  submatrix whose determinant is not zero. Consider function  $\phi : \Theta \to \mathbb{R}$  defined as the determinant of the corresponding  $\varrho \times \varrho$  submatrix of  $\Delta(\theta)$ . We have that  $\phi(\cdot)$  is a polynomial function and is not identically zero on  $\Theta$  since by the construction  $\phi(\theta^*) \neq 0$ . Since  $\Theta$  is connected, it follows that the set  $\{\theta \in \Theta : \phi(\theta) = 0\}$  is "thin", in particular has Lebesgue measure zero. That is,  $\phi(\theta) \neq 0$  and hence  $\operatorname{rank}(\Delta(\theta)) \geq \varrho$  for a.e.  $\theta \in \Theta$ . Also by the definition of  $\varrho$  we have that  $\operatorname{rank}(\Delta(\theta)) \leq \varrho$  for all  $\theta \in \Theta$ . It follows that  $\operatorname{rank}(\Delta(\theta)) = \varrho$  for a.e.  $\theta \in \Theta$ . Since rank of  $\Delta(V, W, X)$  is the same for all  $X \in \mathbb{V}_{\Omega^c}$ , this completes the proof of the assertion (i). Since  $\operatorname{rank}(\Delta(\cdot))$  is a lower semicontinuous function, the assertion (ii) follows.

Now consider a regular point  $\bar{\theta} = (\bar{V}, \bar{W}, \bar{X})$  with  $\bar{X} = 0$ , and the corresponding matrix  $\bar{Y} = \bar{V}\bar{W}^{\top}$ . Since  $\bar{\theta}$  is regular, we have that rank of  $\Delta(\theta)$  is constant (equal  $\varrho$ ) for all  $\theta$  in a neighborhood of  $\bar{\theta}$ . By the Constant Rank Theorem it follows that there is a neighborhood  $\mathcal{V}$  of  $\bar{\theta}$  such that the set  $\mathcal{S} := \{\mathfrak{F}(\theta) :$  $\theta \in \mathcal{V}\}$  forms a smooth manifold of dimension  $\varrho$  in  $\mathbb{R}^{n_1 \times n_2}$ . The tangent space to this manifold at  $\bar{Y}$  is the space  $\mathcal{T}_{\mathcal{M}_r}(\bar{Y}) + \mathbb{V}_{\Omega^c}$ . Hence if  $\varrho = \mathfrak{f}(r, m)$ , then

$$\dim \left( \mathcal{T}_{\mathcal{M}_r}(\bar{Y}) + \mathbb{V}_{\Omega^c} \right) = \dim (\mathcal{T}_{\mathcal{M}_r}(\bar{Y})) + \dim (\mathbb{V}_{\Omega^c}).$$

Consequently dim  $(\mathcal{T}_{\mathcal{M}_r}(\bar{Y}) \cap \mathbb{V}_{\Omega^c}) = 0$ , and thus condition (21) follows (compare with Proposition III.1). On the other hand if  $\rho < \mathfrak{f}(r,m)$ , then the manifold  $(\mathbb{V}_{\Omega^c} + \bar{Y}) \cap \mathcal{M}_r$ , in a neighborhood of  $\bar{Y}$ , has a positive dimension. Thus in that case the solution of MRMC is not locally unique and condition (21)

does not hold. This completes the proof of the assertions (iii) and (iv).

*Proof of Theorem III.5:* Suppose that  $\Omega$  is reducible. Then by making permutations of rows and columns if necessary, it can be assumed that M has the block diagonal form as in (27). Let  $\overline{Y}$  be a respective minimum rank solution. That is  $M_1 = V_1 W_1^{\top}, \ M_2 = V_2 W_2^{\top} \text{ and } \bar{Y} = V W^{\top} \text{ with } V = \binom{V_1}{V_2}$ and  $W = \binom{W_1}{W_2}$  being  $n_1 \times r$  and  $n_2 \times r$  matrices of rank r. Note that  $\bar{Y} = \begin{pmatrix} M_1 & V_1 W_2^\top \\ V_2 W_1^\top & M_2 \end{pmatrix}$ . By changing  $V_1$  to  $\alpha V_1$  and  $W_1$  to  $\alpha^{-1}W_1$  for  $\alpha \neq 0$ , we change matrix  $\bar{Y}$  to matrix  $\binom{M_1}{\alpha^{-1}V_2W_1^{\top}} \stackrel{\alpha V_1W_2^{\top}}{M_2}$ . If  $V_1W_2^{\top} \neq 0$  or  $V_2W_1^{\top} \neq 0$ , we obtain that solution  $\overline{Y}$  is not locally unique. On the other hand when both  $V_1 W_2^{\top} = 0$  and  $V_2 W_1^{\top} = 0$ , and hence  $\bar{Y} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$ , rank r solutions for example are matrices of the form  $\bar{Y} = \begin{pmatrix} M_1 & 0 \\ M_3 & M_2 \end{pmatrix}$ , where columns of matrix  $M_3$  are linear combinations of columns of matrix  $M_1$ . If  $M_1 = 0$ , then we can use matrix  $\overline{Y} = \begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix}$ in the similar way. Hence nonuniqueness of rank r solutions follows

*Proof of Theorem III.6:* Suppose that  $\Omega$  is irreducible. Consider a rank one solution  $\overline{Y} = vw^{\top}$  with respective vectors  $v = (v_1, ..., v_{n_1})^{\top}$  and  $w = (w_1, ..., w_{n_2})^{\top}$ . We can assume that  $v_1$  is fixed, say  $v_1 = 1$ . Consider an element  $M_{1j_1}$ ,  $(1, j_1) \in \Omega$ , in the first row of matrix M. Since it is assumed that each row has at least one observed entry, such element exists. Since  $M_{1j_1} = v_1 w_{j_1}$ , it follows that the component  $w_{j_1}$  of vector w is uniquely defined. Next consider element  $M_{i_1,j_1}$ ,  $(i_1,j_1) \in \Omega$ . Since  $M_{i_1j_1} = v_{i_1}w_{j_1}$ , it follows that the component  $v_{i_1}$  of vector v is uniquely defined. We proceed now iteratively. Let  $\nu \in \{1, ..., n_1\}$  and  $\omega \in \{1, ..., n_2\}$  be index sets for which the respective components of vectors v and w are already uniquely defined. Let  $j \notin \omega$  be such that there is  $(i, j') \in \Omega$  with  $j' \in \omega$ and hence  $w_{i'}$  is already uniquely defined. Since  $M_{ij} = v_i w_j$ and  $M_{ij'} = v_i w_{j'}$ , it follows that  $w_j$  is uniquely defined and j can be added to the index set  $\omega$ . If such column j does not exist, take row  $i \notin \nu$  such that there is  $(i', j) \in \Omega$  with  $i' \in \nu$ . Then  $v_i$ is uniquely defined and hence *i* can be added to  $\nu$ . Since  $\Omega$  is irreducible, this process can be continued until all components of vectors v and w are uniquely defined.

*Proof of Proposition III.3:* Consider function defined in (34). The differential of f(Y) can be written as

$$df(Y) = tr[(P_{\Omega}(Y) - M)^{\top} dY].$$

Therefore if  $Y \in \mathcal{M}_r$  is an optimal solution of the least squares problem (5), then  $\nabla f(Y) = P_{\Omega}(Y) - M$  is orthogonal to the tangent space  $T_{\mathcal{M}_r}(Y)$ . By (20) this implies optimality conditions (32).

Proof of Proposition III.4: Consider function  $\phi$  defined in (33), and the problem of minimization of  $\phi(Y, \Theta)$  subject to  $Y \in \mathcal{M}_r$  with  $\Theta$  viewed as a parameter. Locally for Y near  $\overline{Y} \in \mathcal{M}_r$  the manifold  $\mathcal{M}_r$  can be represented by a system of  $K = n_1 n_2 - \dim(\mathcal{M}_r)$  equations  $g_i(Y) = 0, i = 1, ..., K$ , for an appropriate smooth mapping  $g = (g_1, ..., g_K)$ . That is, the above optimization problem can be written as

$$\min \phi(y, \theta)$$
 subject to  $g_i(y) = 0, \ i = 1, ..., K,$  (45)

where with some abuse of the notation we write this in terms of vectors y = vec(Y) and  $\theta = \text{vec}(\Theta)$ . Note that the mapping g is such that the gradient vectors  $\nabla g_1(\bar{y}), ..., \nabla g_K(\bar{y})$  are linearly independent.

First order optimality conditions for problem (45) are

$$\nabla_y L(y, \lambda, \theta) = 0, \ g(y) = 0, \tag{46}$$

where  $L(y, \lambda, \theta) := f(y, \theta) + \lambda^{\top} g(y)$  is the corresponding Lagrangian. For  $\theta = \theta_0$  this system has solution  $\bar{y}$  and the corresponding vector  $\bar{\lambda} = 0$  of Lagrange multipliers. We can view (46) as a system of (nonlinear) equations in  $z = (y, \lambda)$  variables.

We would like now to apply the Implicit Function Theorem to this system of equations to conclude that for all  $\theta$  near  $\theta_0$ it has unique solution near  $\bar{z} = (\bar{y}, \bar{\lambda})$ . Consider the Jacobian matrix  $\begin{pmatrix} H & G^{\top} \\ G & 0 \end{pmatrix}$  of the system (46) at  $(y, \lambda) = (\bar{y}, \bar{\lambda})$ , where  $H := \nabla_{yy} \phi(\bar{y}, \theta_0)$  is the Hessian matrix of the objective function and  $G := \nabla g(\bar{y}) = [\nabla g_1(\bar{y}), ..., \nabla g_K(\bar{y})]$ . We need to verify that this Jacobian matrix is nonsingular. This is implied by condition (21), which is equivalent to condition (35). Indeed suppose that

$$\begin{bmatrix} H & G \\ G^{\top} & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = 0, \tag{47}$$

for some vectors v and u of appropriate dimensions. This means that Hv + Gu = 0 and  $G^{\top}v = 0$ . It follows that  $v^{\top}Hv = 0$ . Condition  $G^{\top}v = 0$  means that v is orthogonal to the tangent space  $\mathcal{T}_{\mathcal{M}_r}(\bar{y})$ . It follows then by condition (35) that v = 0. Then Gu = 0 and hence, since G has full column rank, it follows that u = 0. Since equations (47) have only zero solution, it follows that this Jacobian matrix is nonsingular. Now by implying the Implicit Function Theorem to the system (46) we obtain the required result. This completes the proof.

Proof of Proposition IV.2: Note that under the specified assumptions,  $M_{ij} - Y_{ij}^*$  are of stochastic order  $O_p(N^{-1/2})$ . We have by Proposition IV.1 that an optimal solution of problem (37) converges in probability to  $Y^*$ . By the standard theory of least squares (e.g., [41, Lemma 2.2]) we can write the following local approximation near  $Y^*$  as (40). It follows that the limiting distribution of  $T_N(r)$  is the same as the limiting distribution of N times the first term in the right hand side of (40). Note that  $N^{1/2} w_{ii}^{1/2} E_{ij}$  converges in distribution to normal with mean  $\sigma_{ij}^{-1}\Delta_{ij}$  and variance one. It follows that the limiting distribution of N times the first term in the right hand side of (40), and hence the limiting distribution of  $T_N(r)$ , is noncentral chisquare with degrees of freedom  $\nu = m - \dim (P_{\Omega}(\mathcal{L}))$  and the noncentrality parameter  $\delta_r$ . Recall that dimension of the linear space  $\mathcal{L}$  is equal to the sum of the dimension of its image  $P_{\Omega}(\mathcal{L})$ plus the dimension of the kernel  $\operatorname{Ker}(P_{\Omega})$ . It remains to note that condition (21) means that  $\operatorname{Ker}(P_{\Omega}) = \{0\}$  (see Remark III.4), and hence

$$\dim \left( P_{\Omega}(\mathcal{L}) \right) = \dim \left( \mathcal{L} \right) = r(n_1 + n_2 - r).$$
(48)

This completes the proof.

*Justification for Theorem III.7:* Note that for both problems (28) and (29) the Slater condition holds, and hence there is no duality gap between these problems, and both problems have

nonempty bounded sets of optimal solutions. Optimality conditions (necessary and sufficient) for problem (9) are

$$C = P_{\mathfrak{S}^c}(\Lambda),\tag{49}$$

$$(\Xi + X)\Lambda = 0, (50)$$

$$\Lambda \succeq 0, \ \Xi + X \succeq 0, \ X \in \mathbb{W}_{\mathfrak{S}^c}.$$
(51)

Now suppose that  $\bar{X} \in \mathbb{W}_{\mathfrak{S}^c}$  is such that  $\Xi + \bar{X} \succeq 0$  and  $\operatorname{rank}(\Xi + \overline{X}) = r < p$ . Let E be a  $p \times (p - r)$  matrix of rank p-r such that  $(\Xi + \overline{X})E = 0$ . By the optimality conditions (49)–(51) we have that  $\overline{X}$  is an optimal solution of the SDP problem (9) if and only if the following condition holds: there exists  $Z \in \mathbb{S}^{p-r}_+$  such that  $P_{\mathfrak{S}^c}(EZE^{\top}) = C$ . Equations  $P_{\mathfrak{S}^c}(EZE^{\top}) = C$  can be viewed as a system of dim $(\mathbb{W}_{\mathfrak{S}^c})$ equations with (p-r)(p-r+1)/2 unknowns (nonduplicated elements of matrix  $Z \in \mathbb{S}^{p-r}$ ). When r is "small" and consequently  $(p-r)(p-r+1)/2 > \dim(\mathbb{W}_{\mathfrak{S}^c})$ , it is likely that this system will have a solution  $Z \succeq 0$ , and hence  $\overline{X}$  is an optimal solution of problem (9). We can also view this by adjusting weight matrix C to the considered matrix  $\Xi + X$  by choosing  $Z \succeq 0$ and defining  $C := P_{\mathfrak{S}^c}(EZE^{\top})$ . For such C the corresponding SDP problem has  $\bar{X}$  as an optimal solution. Note that although matrix  $EZE^{\top}$  is positive semidefinite when  $Z \succeq 0$ , there is no guarantee that the corresponding matrix  $P_{\mathfrak{S}^c}(EZE^{\top})$  is positive semidefinite.

#### REFERENCES

- E. J. Candès and B. Recht, "Exact matrix completion via convex optimization," *Found. Comput. Math.*, vol. 9, no. 6, pp. 717–772, 2009.
- [2] B. Recht, M. Fazel, and P. A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," *SIAM Rev.*, vol. 52, no. 3, pp. 471–501, 2010.
- [3] E. J. Candès and T. Tao, "The power of convex relaxation: Near-optimal matrix completion," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2053– 2080, May 2010.
- [4] M. Davenport and J. Romberg, "An overview of low-rank matrix recovery from incomplete observations," *IEEE J. Sel. Topics Signal Process.*, vol. 10, no. 4, pp. 608–622, Jun. 2016.
- [5] M. Fazel, "Matrix rank minimization with applications," Ph.D. dissertation, Stanford Univ., Stanford, CA, USA, 2002.
- [6] Y.-P. Hsieh, Y.-C. Kao, R. Mahabadi, Y. Alp, and A. Kyrillidis, "A non-Euclidean gradient descent framework for non-convex matrix factorization," *IEEE Trans. Signal Process.*, vol. 66, no. 22, pp. 5917–5926, Nov. 2018.
- [7] Y. C. Eldar, D. Needell, and Y. Plan, "Uniqueness conditions for lowrank matrix recovery," *Appl. Comput. Harmon. Anal.*, vol. 33, no. 2, pp. 309–314, Sep. 2012.
- [8] D. Pimentel-Alarcon, N. Boston, and R. D. Nowak, "A characterization of deterministic sampling patterns for low-rank matrix completion," *IEEE J. Sel. Topics Signal Process.*, vol. 10, no. 4, pp. 623–636, Jun. 2016.
- [9] M. Ashraphijuo, V. Aggarwal, and X. Wang, "On deterministic sampling patterns for robust low-rank matrix completion," *IEEE Signal Process. Lett.*, vol. 25, no. 3, pp. 343–347, Dec. 2017.
- [10] D. L. Pimentel-Alarcón and R. D. Nowak, "A converse to low-rank matrix completion," in *Proc. IEEE Int. Symp. Inf. Theory*, 2016, pp. 96–100.
- [11] M. Ashraphijuo, X. Wang, and V. Aggarwal, "Rank determination for lowrank data completion," *J. Mach. Learn. Res.*, vol. 18, no. 98, pp. 1–29, Sep. 2017.
- [12] M. Ashraphijuo and X. Wang, "Fundamental conditions for low-CP-rank tensor completion," J. Mach. Learn. Res., vol. 18, no. 63, pp. 1–29, Jul. 2017.
- [13] B. I. Dundas, A Short Course in Differential Topology. Cambridge, U.K.: Cambridge Univ. Press, 2018.
- [14] A. Shapiro, "Statistical inference of semidefinite programming," *Math. Program.*, pp. 1–21, 2018.

- [15] J. A. Tropp, A. Yurtsever, M. Udell, and V. Cevher, "Practical sketching algorithms for low-rank matrix approximation," *SIAM J. Matrix Anal. Appl.*, vol. 38, no. 4, pp. 1454–1485, Dec. 2017.
- [16] R. Sun and Z.-Q. Luo, "Guaranteed matrix completion via non-convex factorization," *IEEE Trans. Inf. Theory*, vol. 62, no. 11, pp. 6535–6579, 2014.
- [17] C. Ma, K. Wang, Y. Chi, and Y. Chen, "Implicit regularization in nonconvex statistical estimation: Gradient descent converges linearly for phase retrieval, matrix completion and blind deconvolution," 2017, arXiv:1711.10467.
- [18] E. Candés and B. Recht, "Exact matrix completion via convex optimization," *Found. Comput. Math.*, vol. 9, pp. 717–772, 2009.
- [19] E. J. Candes and T. Tao, "The power of convex relaxation: Near-optimal matrix completion," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2053– 2080, May 2010.
- [20] B. Recht, "A simpler approach to matrix completion," J. Mach. Learn. Res., vol. 12, pp. 3414–3430, 2011.
- [21] D. Gross, "Recovering low-rank matrices from few coefficients in any basis," *IEEE Trans. Inf. Theory*, vol. 57, no. 3, pp. 1548–1566, Mar. 2011.
- [22] Y. Chen, "Incoherence-optimal matrix completion," *IEEE Trans. Inf. Theory*, vol. 61, no. 5, pp. 2909–2923, May 2014.
- [23] Y. Chen, S. Bhojanapalli, S. Sanghavi, and R. Ward, "Coherence matrix completion," in *Proc. Int. Conf. Mach. Learn.*, 2014, pp. 1881–1889.
- [24] M. W. Browne, "Statistical inference in factor analysis," in *Topics in Applied Multivariate Analysis*, D. M. Hawkins, Ed. Cambridge, U.K.: Cambridge Univ. Press, 1982.
- [25] J.-F. Cai, E. J. Candes, and Z. Shen, "A singular value thresholding algorithm for matrix completion," *SIAM J. Optim.*, vol. 20, no. 4, pp. 1956– 1982, 2010.
- [26] G. Pataki, "On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues," *Math. Operations Res.*, vol. 23, pp. 339–358, 1998.
- [27] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, "The convex geometry of linear inverse problems," *Found. Comput. Math.*, vol. 12, pp. 805–849, 2012.
- [28] A. Shapiro, "Asymptotic theory of overparameterized structural models," J. Amer. Statistical Assoc., vol. 81, pp. 142–149, 1986.
- [29] T. W. Anderson and H. Rubin, "Statistical inference in factor analysis," in Proc. 3rd Berkeley Symp. Math. Statist. Probability, J. Neyman, Ed. Univ. of California Press, 1956, pp. 111–150.
- [30] E. Wilson and J. Worcester, "The resolution of six tests into three general factors," *Proc. Nat. Acad. Sci. USA*, vol. 25, pp. 73–77, 1939.
- [31] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, 3rd ed. Cambridge, MA, USA: MIT Press, 2009.
- [32] A. Shapiro, "Extremal problems on the set of nonnegative definite matrices," *Linear Algebra Appl.*, vol. 67, pp. 7–18, 1985.
- [33] A. Shapiro, "Rank reducibility of a symmetric matrix and sampling theory of minimum trace factor analysis," *Psychometrika*, vol. 47, pp. 187–199, 1982.
- [34] A. Shapiro, "Weighted minimum trace factor analysis," *Psychometrika*, vol. 47, pp. 243–264, 1982.
- [35] P. Bekker and J. Ten Berge, "Generic global indentification in factor analysis," *Linear Algebra Appl.*, vol. 264, pp. 255–263, 1997.
- [36] D. McManus, "Who invented local power analysis?" *Econometric Theory*, vol. 7, pp. 265–268, 1991.
- [37] J. Steiger, A. Shapiro, and M. Browne, "On the multivariate asymptotic distribution of sequential chi-square statistics," *Psychometrika*, vol. 50, pp. 253–254, 1985.
- [38] S. R. Becker, E. J. Candès, and M. C. Grant, "Templates for convex cone problems with applications to sparse signal recovery," *Math. Program. Comput.*, vol. 3, no. 3, pp. 165–218, 2011.

- [39] R. Mazumder, T. Hastie, and R. Tibshirani, "Spectral regularization algorithms for learning large incomplete matrices," *J. Mach. Learn. Res.*, vol. 11, no. Aug, pp. 2287–2322, 2010.
- [40] R. H. Keshavan, A. Montanari, and S. Oh, "Matrix completion from a few entries," *IEEE Trans. Inf. Theory*, vol. 56, no. 6, pp. 2980–2998, Jun. 2010.
- [41] A. Shapiro, "Asymptotic distribution of test statistics in the analysis of moment structures under inequality constraints," *Biometrika*, vol. 72, pp. 133–144, 1985.



Alexander Shapiro is currently a Professor with the School of Industrial and Systems Engineering, with joined appointment in the Mathematics Department, Georgia Institute of Technology, Atlanta, GA, USA. He has authored more than 140 research articles in peer review journals and has coauthored of several books. He was on the editorial board of several professional journals, he was an Area Editor (Optimization) of Operations Research, and the Editor-in-Chief of the *Mathematical Programming*, Series A, journal. He gave numerous invited keynote and plenary talks,

including invited section talk (section Control Theory & Optimization) at the International Congress of Mathematicians 2010, Hyderabad, India. In 2013, he was the recipient of Khachiyan Prize for Life-time Accomplishments in Optimization, awarded by the INFORMS Society, and in 2018 he was also the recipient of the George B. Dantzig Prize awarded by the Mathematical Optimization Society and Society for Industrial and Applied mathematics.



Yao Xie received the B.Sc. in electrical engineering and computer science from the University of Science and Technology of China, Hefei, China, in 2004, the M.Sc. degree in electrical and computer engineering from the University of Florida, Gainesville, FL, USA, in 2006, and the Ph.D. degree in electrical engineering (minor in Mathematics) from Stanford University, Stanford, CA, USA, in 2011. She is an Assistant Professor with the H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA, USA. From 2012 to 2013, she

was a Research Scientist with Duke University. Her research areas include statistics, signal processing, and machine learning, in providing theoretical foundation as well as developing computationally efficient and statistically powerful algorithms for big data in various applications such as sensor networks, imaging, and crime data analysis. She was the recipient of the National Science Foundation CAREER Award in 2017.



**Rui Zhang** received the B.S. and M.S. degrees in statistics from Sun Yat-sen University, Guangzhou, China and the University of Michigan, Ann Arbor, MI, USA, in 2015 and 2017, respectively. He is currently working toward the Ph.D. degree with the H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA, USA. He is interested in change-point detection, machine learning, and statistics.